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GROWTH RATES OF MOMENT SEQUENCES

By

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Growth Rates of Moment Sequences

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1. Introduction.

Let X be a non-negative random variable having distribution function F such that its nth moment μ_n' about the origin is finite for all positive integers. Then $\{\mu_n'\}$ is a moment sequence associated with F. The study of moment sequences has received considerable attention (see, e.g., [1], [6], and [10]), including determination of criteria (a) for a sequence of real numbers to be a moment sequence and (b) for the sequence $\{\mu_n'\}$ to determine F uniquely. Interest here is focused on the behavior of the ratio μ_{n+1}'/μ_n' of moments as $n \to +\infty$ and, in particular, the construction of distribution functions having a "suitably" prescribed growth rate.

It is well-known [10] that a sequence of real numbers $1, \mu_1', \mu_2', \ldots,$ is a moment sequence of a non-negative random variable if and only if the quadratic forms

$$(1.1) \qquad Q_0 = \sum_{i,j=0}^{n} \mu_{i+j}' \, x_i \, x_j \quad \text{and} \quad Q_1 = \sum_{i,j=0}^{n} \mu_{i+j+1}' \, x_i \, x_j$$

are positive semi-definite for all $n \geq 0$ and for every sequence of real numbers $x_0, x_1, \ldots, x_n$. An elementary consequence of (1.1) is that the sequence $\{\mu_{n+1}'/\mu_n'\}$ is non-decreasing. If F is a finite distribution function (i.e., $b = \inf\{x : F(x) = 1\} < \infty$) then $\lim(\mu_n')^{1/n} = b$

(see Boas [2]). By a well-known theorem this implies that $\lim \mu'_{n+1}/\mu'_n = b$. Consequently, the study of the growth rate of the ratio μ'_{n+1}/μ'_n for finite distributions is "uninteresting".

If the support of F is unbounded, i.e., $F(x) < 1$ for all $x > 0$, then $\lim \mu'_{n+1}/\mu'_n = +\infty$. To see this, suppose that the limit L is finite. Then $(\mu'_n)^{1/n} \to L$. Let $A > 0$ be an arbitrary but fixed number. Then

$$\mu'_n = \int_0^\infty x^n \, dF(x) \geq \int_A^\infty x^n \, dF(x) \geq A^n[1 - F(A)]$$

and hence $\liminf \mu_n^{1/n} \geq A$. Since A was arbitrary, we conclude that $(\mu'_n)^{1/n} \to +\infty$, a contradiction.

Thus, we see that distributions of non-negative random variables with finite and unbounded supports may be distinguished in terms of the behavior of the ratio of their $(n+1)$st and nth moments. Further consideration of the study of such ratios is in the direction of exhibiting distributions of non-negative random variables for which the ratio asymptotically behaves in a suitably prespecified manner. Before proceeding, however, we give several examples which illustrate the behavior of moment ratios and "suggest" how to carry out the construction just referred to.

<u>Example 1</u>. $F(x) = \begin{Bmatrix} 0 & x \leq 0 \\ 1-p & 0 < x \leq C \\ 1 & x > C \end{Bmatrix}$.

Here, $\mu'_n = pC^n$ and $\mu'_{n+1}/\mu'_n = C$. For $0 < C < 1$, $C = 1$, $C > 1$, $\mu'_n \to$ $\to 0, 1, +\infty$, the ratio is equal to C independently of n.

<u>Example 2</u>. $F(x) = \begin{Bmatrix} 0 & x \leq 0 \\ x/C & 0 \leq x \leq C \\ 1 & x \geq C \end{Bmatrix}$.

In this case, $\mu_n' = C^n/(n+1)$, $\mu_{n+1}'/\mu_n' = (n+1)C/(n+2) \to C$; as in Example 1, $\mu_n' \to 0, 1,$ or $+\infty$, according to whether $C < 1$, $C = 1$, or $C > 1$.

Example 3. $F(x) = \begin{Bmatrix} 0 & x \leq 0 \\ \int_0^x y^{\alpha-1} e^{-y/\beta} dy/\Gamma(\alpha)\beta^\alpha \end{Bmatrix}$.

It is easily seen that $\mu_n' = \Gamma(n+\alpha)\beta^n/\Gamma(\alpha)$ and $\mu_{n+1}'/\mu_n' = (n+\alpha)\beta$; also, $[\mu_{n+1}'/\mu_n']/n \to \beta$.

Example 4. $F(x) = \begin{Bmatrix} 0 & x \leq 0 \\ 1 - e^{-x^\beta}, & x > 0 \end{Bmatrix}$.

A simple calculation shows that $\mu_n' = \Gamma(\frac{n+\beta}{\beta})$ and $\mu_{n+1}'/\mu_n' \simeq (\frac{n}{\beta}+1)^{1/\beta}$. Thus, $\mu_{n+1}'/n^{1/\beta}\,\mu_n' \to C$, $0 < C < \infty$. In particular, if $\beta = 1/k$, k an integer, then $\mu_{n+1}'/n^k\,\mu_n' \to k^k$. If $\beta = 1/2$, $\mu_n' = \Gamma(2n+1) = (2n)!$ for $n \geq 1$, while if $\beta = 1/4$, $\mu_n' = \Gamma(4n+1) = (4n)!$ for $n \geq 1$. If $\beta = 1/2$, then $\mu_{n+1}'/\mu_n' \simeq 4n^2$ while $\mu_{n+1}'/\mu_n' \simeq 256n^4$ for $\beta = 1/4$.

Example 5. $F(x) = \begin{Bmatrix} 0 & x \leq 1 \\ 1 - e^{-(\ln x)^2}, & x \geq 1 \end{Bmatrix}$.

It can be shown that $\mu_n' \simeq e^{n^2/4}$, $\mu_{n+1}'/\mu_n' \simeq e^{(2n+1)/4}$.

Example 6. Let X be normally distributed with mean zero and variance $\alpha = \sigma^2/2$ and put $Y = e^X$, so that Y has a log-normal distribution. Then it can easily be shown that $\mu_n' = E(Y^n) = e^{\alpha n^2}$ and $\mu_{n+1}'/\mu_n' = e^{(2n+1)\alpha}$. Thus, asymptotically, the distribution in Example 5 and a log-normal random variable with parameter $\alpha = 1/4$ (corresponding to $\sigma^2 = 1/2$) have moments such that the ratios of the (n+1)st and nth moments are equal.

2. Assumptions and Statement of Results.

Examples 3, 4, and 5 illustrate the fact that the more "slowly" $1-F(x) \to 0$ as $x \to +\infty$, the more "rapidly" μ'_{n+1}/μ'_n tends to $+\infty$ with n. The strength of this "inverse relationship" between $1-F(x)$ and μ'_{n+1}/μ'_n is reflected in the examples. Thus, if $-\ln[1-F(x)] = x^\beta$, then $\mu'_{n+1}/\mu'_n \simeq [(n+\beta)/\beta]^{1/\beta}$, while if $-\ln[1-F(x)] = (\ln x)^2$, then $\mu'_{n+1}/\mu'_n \simeq e^{(2n+1)/4}$.

We assume that the distribution function behaves in a "smooth" manner. Explicitly, we assume that the log of the "tail" of F is the inverse of either a regularly varying or a rapidly varying function (but not "too" rapid). Before stating our assumptions and results, we recall the definition of "regularly" varying functions and list several properties. For a complete exposition on the subject, see Seneta [7].

Definition 2.1. A function ϕ is said to be regularly varying at infinity if it is real-valued, positive and measurable on $[A,\infty)$, for some $A > 0$, and if for each $x > 0$,

$$\lim_{t\to\infty} \phi(tx)/\phi(t) = x^\rho$$

for some ρ in the interval $-\infty < \rho < \infty$; ρ is called the index of regular variation.

Definition 2.2. A function L which is regular varying, with index of regular variation $\rho = 0$, is called slowly varying.

Every regularly varying function ϕ is necessarily of the form $\phi(x) = x^\rho L(x)$, with L slowly varying. For slowly varying functions, we have the following results (from Seneta [7]):

(2.1) $\lim_{t\to\infty} L(tx)/L(t) = 1$ uniformly for x in closed intervals $[a,b]$, $0 < a < b < \infty$.

(2.2) For any $\gamma > 0$, $x^{\gamma} L(x) \to +\infty$ and $x^{-\gamma} L(x) \to 0$, as $x \to \infty$.

(2.3) $\ln L(x)/\ln x \to 0$ as $x \to \infty$.

(2.4) If L, defined on $[A,\infty)$, $A > 0$, is slowly varying, then there exists a positive number $B \geq A$ such that for all $x \geq B$,

$$L(x) = \exp\left\{\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}\, dt\right\}$$

where η is a bounded measurable function on $[B,\infty)$ such that $\eta(x) \to C$ finite, and ε is a continuous function on $[B,\infty)$ such that $\varepsilon(x) \to 0$ as $x \to \infty$. Conversely, any function L having the above representation where η and ε have the properties stated is slowly varying.

(2.5) If L is a slowly varying function which is eventually non-decreasing (non-increasing), then the continuous $\varepsilon(t)$ in its representation for sufficiently large values may be taken as satisfying $\varepsilon(t) \geq 0$ (≤ 0).

(2.6) For a given slowly varying function L on $[A,\infty)$, there exists another, infinitely differentiable, slowly varying function L_1 with the following properties:

(a) $L_1(x) \sim L(x)$, as $x \to \infty$;

(b) $L_1(n) = L(n)$ for all integers n sufficiently large;

(c) If L is ultimately monotone, then so is L_1;

(d) If L is ultimately convex, then so is L_1.

(2.7) If L_1 and L_2 are slowly varying, then so are $L_1 L_2$, $L_1 + L_2$ and L_i^{α} for $\alpha \neq 0$, $i = 1, 2$.

(2.8) If ϕ is a regularly varying differentiable function of index ρ, then $x\,\phi'(x)/\phi(x) = x\frac{d}{dx}\ln\phi(x) \to \rho$ as $x \to \infty$.

The results which we present here are generalizations of Example 3 and 4. Explicitly, we consider two cases:

(i) $-\ln[1-F(x)] = \phi^{-1}(x)$, where ϕ is a monotonically increasing twice differentiable regularly varying function of index $\rho \geq 0$, so that $\phi(x) = x^{\rho}L(x)$; we assume $\ln L(x) = \int_1^x [\varepsilon(y)/y]dy$, where ε is also assumed to be slowly varying. If $\rho = 0$, $\ln L(x)$ increases to $+\infty$ as $x \to +\infty$; in this case, $\varepsilon(x) > 0$ and decreases to zero. In order for F to be a distribution function, $\phi^{-1}(x) \to +\infty$, so that $\phi(x)$ also must increase to $+\infty$.

(ii) $-\ln[1-F(x)] = \lambda^{-1}(\ln x)$, where λ is a monotonically increasing twice differentiable regularly varying function of index $0 \leq \rho \leq 1$. Thus, $\lambda(x) = x^{\rho}L(x)$, with $0 \leq \rho \leq 1$. If $\rho = 0$, then $L(x) \uparrow +\infty$, so that the $\varepsilon(x)$ in the representation of L is a positive function decreasing to zero as $x \to +\infty$. If $0 < \rho < 1$, then $\varepsilon(x)$ is either strictly monotone or is taken to be identically zero. Finally, if $\rho = 1$, then $L(x)$ decreases to zero as $x \to \infty$, so that $\varepsilon(x) < 0$. It is assumed also that $\varepsilon(x)$ is slowly varying. If $\rho = 0$, we assume $\phi(x)$ varies rapidly, to avoid overlap with case (i).

We prove the following two theorems, corresponding to the two cases.

Theorem 1. In case (i), there exists $\{\tau_n\}$, satisfying

(2.9) $$\tau_n = n[\rho + \varepsilon(\tau_n)]$$

such that

(2.10) $$\lim_{n\to\infty} \mu_n'/\sqrt{2\pi\ \tau_n}\ g_n(\tau_n) = 1$$

where

$$\ln g_n(\tau_n) = n \ln \phi(\tau_n) - \tau_n ;$$

(2.11) $$\lim_{n\to\infty} \mu_{n+1}'/\phi(\tau_n)\mu_n' = 1 .$$

<u>Theorem 2</u>. In case (ii), there exists $\{\tau_n\}$ satisfying

(2.12) $$\tau_n = n[\rho + \varepsilon(\tau_n)]\ \lambda(\tau_n)$$

such that

(2.13) $$\lim_{n\to\infty} \mu_n'/\sqrt{2\pi\ \tau_n/(1-\rho)}\ g_n(\tau_n) = 1 \quad \text{if} \quad 0 \leq \rho < 1$$

(2.14) $$\lim_{n\to\infty} \mu_n'/\sqrt{-2\pi\ \tau_n/\varepsilon(\tau_n)}\ g_n(\tau_n) = 1 \quad \text{if} \quad \rho = 1 .$$

with $\ln g_n(\tau_n) = n\ \lambda(\tau_n) - \tau_n$.

(2.15) $$\lim_{n\to\infty}\ \ln[\mu_{n+1}'/\mu_n']/\lambda(\tau_{n+1}) = 1 , \qquad \text{if} \quad 0 \leq \rho < 1$$

or if $\rho = 1$ and $\tau_{n+1}/\tau_n \to 1$.

(2.16) $$\lim_{n\to\infty}\ \ln[\mu_{n+1}'/\mu_n']/[-\varepsilon(\tau_{n+1})\ \lambda(\tau_{n+1})] \to (L-1)/L , \quad \text{if} \quad \rho = 1$$

and $\tau_{n+1}/\tau_n \to L > 1$. If $\tau_{n+1}/\tau_n \to +\infty$, then the limit in (2.16) is 1.

Examples where τ_{n+1}/τ_n converges to 1, $L(<\infty)$, or $+\infty$ are given.

3. Proof of Theorem 1.

We have $-\ln[1-F(x)] = \phi^{-1}(x)$, so that $F(x) = 1 - e^{-\phi^{-1}(x)}$ and

$$\mu_n' = \int_0^\infty x^n \, dF(x) = n \int_0^\infty x^{n-1}[1-F(x)]dx = n \int_0^\infty x^{n-1} \, e^{-\phi^{-1}(x)}dx \; .$$

Let $y = \phi^{-1}(x)$, so that $x = \phi(y)$. Then

$$\mu_n' = n \int_0^\infty [\phi(y)]^{n-1} \, \phi'(y) \, e^{-y}dy = \int_0^\infty [\phi(y)]^n \, e^{-y}dy \tag{3.1}$$

on integrating by parts. Let

$$\ln g_n(y) = \ln[\phi(y)]^n \, e^{-y} = n\left\{\rho \ln y + \int_1^y \frac{\varepsilon(u)}{u} \, du\right\} - y \; . \tag{3.2}$$

Then

$$\frac{d}{dy} \ln g_n(y) = n[\rho + \varepsilon(y)]/y - 1 \tag{3.3}$$

and

$$\frac{d^2}{dy^2} \ln g_n(y) = - \frac{n}{y^2}\left\{\rho + \varepsilon(y)\left[1 - \frac{y\,\varepsilon'(y)}{\varepsilon(y)}\right]\right\} \; . \tag{3.4}$$

Both $\varepsilon(y)$ and $y\,\varepsilon'(y)/\varepsilon(y) = y \frac{d}{dy} \ln|\varepsilon(y)|$ tend to 0 as $y \to \infty$ (noting (2.8)), so that for y sufficiently large, the right hand side of (3.4) is negative. Thus, the right hand side of (3.3) decreases as

y increases and there exists a unique positive real number τ_n such that $\ln g_n(y)$ attains a maximum at $y = \tau_n$. From (3.3), it is seen that τ_n satisfies $n[\rho + \varepsilon(\tau_n)] = \tau_n$. Furthermore, the sequence $\{\tau_n\}$ increases monotonically since $n[\rho + \varepsilon(y)]/y$ decreases as y increases (if τ_{n+1} were less than τ_n, then we'd have $\frac{1}{n+1} = [\rho + \varepsilon(\tau_{n+1})]/\tau_{n+1} > [\rho + \varepsilon(\tau_n)]/\tau_n = \frac{1}{n}$, a contradiction). Thus, $\tau_n \to L$. Suppose $L < \infty$. Then $\varepsilon(\tau_n) \to \varepsilon(L)$ as $\tau_n \to L$. If $\rho = 0$, we would have $1/n = \varepsilon(\tau_n)/\tau_n \to \varepsilon(L)/L$, which is nonsense. If $\rho > 0$, then $\tau_n/n = \rho + \varepsilon(\tau_n) \to \rho$, so that $\tau_n \simeq n\rho \to +\infty$. Finally, we claim that $\tau_{n+1}/\tau_n \to 1$. If $\rho = 0$, we have

$$0 \leq \tau_{n+1}/\tau_n - 1 = [(n+1)\varepsilon(\tau_{n+1})/n\ \varepsilon(\tau_n)] - 1 \leq (n+1)/n - 1 = \frac{1}{n} \to 0 ,$$

using the fact that $\varepsilon(y) \downarrow 0$ monotonically if $\rho = 0$. If $\rho > 0$, $\tau_{n+1}/\tau_n \simeq [\rho + \varepsilon(\tau_{n+1})]/[\rho + \varepsilon(\tau_n)] \to 1$.

Now let $\delta > 0$ be arbitrary (but less than 1) and write

$$\mu_n' = \int_0^\infty g_n(y)dy = \tau_n \int_0^\infty g_n(\tau_n y)dy = \sum_{j=1}^{4} I_{nj} \tag{3.5}$$

where

$$I_{n1} = \tau_n \int_0^{1-\delta} g_n(\tau_n y)dy , \quad I_{n2} = \tau_n \int_{1-\delta}^{1+\delta} g_n(\tau_n y)dy ,$$

$$I_{n3} = \tau_n \int_{1+\delta}^{4} g_n(\tau_n y)dy , \quad I_{n4} = \tau_n \int_4^\infty g_n(\tau_n y)dy .$$

We shall show that

$$(3.6) \qquad \lim_{n\to\infty} I_{n2}/\sqrt{2\pi\ \tau_n}\ g_n(\tau_n) = 1$$

and

$$(3.7) \qquad \lim_{n\to\infty} I_{nj}/\sqrt{2\pi\ \tau_n}\ g_n(\tau_n) = 0\ , \quad \text{for} \quad j = 1,3,4\ .$$

To prove (3.6), we proceed as follows. Expanding $\ln g_n(\tau_n\ x)$ in a Taylor's series about $x = 1$, we get

$$(3.8) \qquad \begin{aligned} \ln g_n(\tau_n\ x) &= \ln g_n(\tau_n) + \{n[\rho + \varepsilon(\tau_n)] - \tau_n\}(x-1) \\ &\qquad - n\left[\frac{d^2}{dx^2} \ln \phi(\tau_n\ x)\right]_{x=\theta_n} (x-1)^2/2 \\ &= \ln g_n(\tau_n) - n\ r_n(\theta_n)(x-1)^2/2 \end{aligned}$$

where

$$(3.9) \qquad r_n(\theta_n) = [\rho + \varepsilon(\tau_n\ \theta_n)\ S(\tau_n\ \theta_n)]/\theta_n^2$$

with $S(x) = 1 - x\ \varepsilon'(x)/\varepsilon(x) \to 1$ as $x \to \infty$. Thus,

$$(3.10) \qquad I_{n2} = \tau_n\ g_n(\tau_n) \int_{1-\delta}^{1+\delta} \exp\{-n\ r_n(\theta_n)(x-1)^2/2\}dx$$

with $1 - \delta < \theta_n < 1 + \delta$. If $\rho > 0$, we have

$$\frac{n\ r_n(\theta_n)}{\tau_n} \leq \frac{n\rho}{\tau_n(1-\delta)^2} + \frac{|\varepsilon(\tau_n\ \theta_n)\ S(\tau_n\ \theta_n)|}{(1-\delta)^2[\rho+\varepsilon(\tau_n)]}$$

and

$$\frac{n\ r_n(\theta_n)}{\tau_n} \geq \frac{n\rho}{\tau_n(1+\delta)^2} - \frac{|\varepsilon(\tau_n\ \theta_n)\ S(\tau_n\ \theta_n)|}{(1+\delta)^2[\rho+\varepsilon(\tau_n)]}$$

so that

$$(3.11)\qquad (1+\delta)^{-2} \leq \liminf \frac{n\ r_n(\theta_n)}{\tau_n} \leq \limsup \frac{n\ r_n(\theta_n)}{\tau_n} \leq (1-\delta)^{-2}\ .$$

If $\rho = 0$, then

$$\frac{n\ r_n(\theta_n)}{\tau_n} = \frac{n\ \varepsilon(\tau_n\ \theta_n)}{\tau_n\ \theta_n^2}\ S(\tau_n\ \theta_n) \leq \frac{\varepsilon[(1-\delta)\tau_n]}{(1-\delta)^2\ \varepsilon(\tau_n)}\ [1 + |u(\tau_n\ \theta)|]\ .$$

where

$$(3.12)\qquad u(x) = 1 - S(x) = x\ \varepsilon'(x)/\varepsilon(x)\ ,$$

so that $\limsup n\ r_n(\theta_n)/\tau_n \leq (1-\delta)^2$. Similarly, $\liminf n\ r_n(\theta_n)/\tau_n \geq (1+\delta)^{-2}$, so that (3.11) holds for all $\rho \geq 0$. From (3.10) and (3.11) it follows that

$$\frac{1}{(1+\delta)^2} \leq \liminf \frac{I_{n2}}{\sqrt{2\pi\ \tau_n}\ g_n(\tau_n)} \leq \limsup \frac{I_{n2}}{\sqrt{2\pi\ \tau_n}\ g_n(\tau_n)} \leq \frac{1}{(1-\delta)^2}\ .$$

Since δ is arbitrary, (3.6) follows.

As for (3.7), we first show that $I_{n1}/\sqrt{2\pi\,\tau_n}\; g_n(\tau_n) \to 0$. We have

$$\frac{I_{n1}}{\sqrt{2\pi\,\tau_n}\; g_n(\tau_n)} \leq \frac{(1-\delta)\tau_n\; g_n[(1-\delta)\tau_n]}{\sqrt{2\pi\,\tau_n}\; g_n(\tau_n)} = \frac{(1-\delta)}{\sqrt{2\pi}}\, e^{-na_n} \tag{3.13}$$

where

$$n\, a_n = n \int_{(1-\delta)\tau_n}^{\tau_n} \frac{\varepsilon(u)}{u}\, du - \delta\,\tau_n - \frac{1}{2}\ln \tau_n - n\,\rho\,\ln(1-\delta)\ .$$

If $\rho = 0$, then $\varepsilon(u) > 0$ and

$$n\, a_n \geq n\,\varepsilon(\tau_n)\,\ln(1-\delta)^{-1} - \delta\,\tau_n - \frac{1}{2}\ln\tau_n = \tau_n[\ln(1-\delta)^{-1} - \delta] - \frac{1}{2}\ln\tau_n$$

$$> \tau_n\,\delta^2/2 - \ln\tau_n/2 = \tau_n\,\delta^2[1 - (\ln\tau_n)/\tau_n\,\delta]/2 \to +\infty\ .$$

If $\rho > 0$ and $\varepsilon(y) > 0$, then

$$n\, a_n > n\,\varepsilon(\tau_n)\,\ln(1-\delta)^{-1} - \delta\,\tau_n - (\ln\tau_n)/2 - n\,\rho\,\ln(1-\delta)$$

$$= n[\rho + \varepsilon(\tau_n)]\ln(1-\delta)^{-1} - \delta\,\tau_n + (\ln\tau_n)/2$$

$$= \tau_n[\ln(1-\delta)^{-1} - \delta] - (\ln\tau_n)/2 > \tau_n\,\delta^2[1 - (\ln\tau_n)/\tau_n\,\delta^2]/2 \to \infty\ .$$

If $\rho > 0$ and $\varepsilon(y) < 0$, then

$$n\, a_n \geq n\, \rho\, \ln(1-\delta)^{-1} - \delta\, \tau_n - (\ln \tau_n)/2$$

$$= n\{\rho\, \ln(1-\delta)^{-1} - \delta[\rho + \varepsilon(\tau_n)]\} - (\ln \tau_n)/2$$

$$> n\rho\{\ln(1-\delta)^{-1} - \delta\} - (\ln \tau_n)/2$$

$$> n\rho\, \delta^2/2 - (\ln \tau_n)/2 > \tau_n\, \delta^2[1 - (\ln \tau_n)/\tau_n\, \delta^2]/2 \to \infty .$$

Thus, the r.h.s. of (3.13) converges to zero.

Next we show that $I_{n3}/\sqrt{2\pi\, \tau_n}\, g_n(\tau_n) \to 0$. We proceed as in (3.8)-(3.10), but with I_{n2} replaced by I_{n3} and $1 - \delta < \theta_n < 1 + \delta$ replaced by $1 + \delta < \theta_n < 4$. It then is easily seen that for $\rho > 0$,

$$n\, r_n(\theta_n)/\tau_n \geq (n\rho/16\, \tau_n) - |\varepsilon(\tau_n\, \theta_n)\, S(\tau_n\, \theta_n)|/(1+\delta)^2[\rho+\varepsilon(\tau_n)] \to \frac{1}{16}$$

while if $\rho = 0$,

$$n\, r_n(\theta_n)/\tau_n \geq \frac{\varepsilon(4\tau_n)}{16\, \varepsilon(\tau_n)}\, \{1 - |u(\tau_n\, \theta)|\} \to \frac{1}{16} ,$$

using (2.1). Thus, for all n sufficiently large,

$$I_{n3}/\sqrt{2\pi\, \tau_n}\, g_n(\tau_n) \leq \sqrt{\frac{\tau_n}{2\pi}} \int_{1+\delta}^{4} e^{-\tau_n(x-1)^2/64}\, dx \to 0 .$$

Finally, we show that $I_{n4}/\sqrt{2\pi\, \tau_n}\, g_n(\tau_n) \to 0$. But

$$I_{n4}/\sqrt{2\pi\,\tau_n}\; g_n(\tau_n) = \sqrt{\frac{\tau_n}{2\pi}} \int_4^\infty [g_n(\tau_n\, y)/g_n(\tau_n)]dy \;,$$

and

$$\begin{aligned}
\ln[g_n(\tau_n\, y)/g_n(\tau_n)] - \tau_n &= n\left[\rho \ln y + \int_{\tau_n}^{\tau_n y} \frac{\varepsilon(u)}{u}\, du\right] - \tau_n\, y \\
&= \int_{\tau_n}^{\tau_n y} \frac{n[\rho+\varepsilon(u)]}{u}\, du - \tau_n\, y \\
&= \tau_n \left\{\int_{\tau_n}^{\tau_n y} \frac{n[\rho+\varepsilon(u)]}{\tau_n\, u}\, du - y\right\} \\
&= - \tau_n\, y\, k_n(y)
\end{aligned}$$

where

$$k_n(y) = 1 - \frac{n}{\tau_n\, y} \int_{\tau_n}^{\tau_n y} \frac{\rho+\varepsilon(u)}{u}\, du = 1 - \frac{\int_{\tau_n}^{\tau_n y} \frac{\rho+\varepsilon(u)}{u}\, du}{[\rho+\varepsilon(\tau_n)]y} \;.$$

If $\varepsilon(x) > 0$, then $k_n(y) \geq 1 - e^{-1}$, since $1 - k_n(y) \leq [\ln y/y] \leq e^{-1}$ for all y . If $\varepsilon(x) < 0$, then

$$1 - k_n(y) \leq \{\rho \ln y/[\rho+\varepsilon(\tau_n)]y\} < 2 \ln y/y \leq 2\, e^{-1} \;, \text{ for all } y \;,$$

provided we choose n such that $\varepsilon(\tau_n) > -\rho/2$ (which we can do, noting that if $\varepsilon(x) < 0$, then $\rho > 0$ and $\varepsilon(\tau_n) \to 0$). Thus, for all y and n sufficiently large, $k_n(y) \geq 1 - 2\, e^{-1}$. Hence,

$$I_{n4}/\sqrt{2\pi\ \tau_n}\ g_n(\tau_n) < \sqrt{\frac{\tau_n}{2\pi}}\ e^{\tau_n} \int_4^\infty e^{-k\tau_n y} dy = (2\pi\ k\tau_n)^{-1/2}\ e^{(1-4k)\tau_n} \to 0$$

since $1 - 4k < 0$.

This completes the proof of (3.7) and hence of (2.10). To prove (2.11), we have

$$\mu'_{n+1}/\mu'_n\ \phi(\tau_n) = (\tau_{n+1}/\tau_n)^{1/2}\ [g_{n+1}(\tau_{n+1})/g_n(\tau_n)\ \phi(\tau_n)] = A_n\ e^{B_n}$$

with $A_n = (\tau_{n+1}/\tau_n)^{1/2} \to 1$ and

$$B_n = \ln[g_{n+1}(\tau_{n+1})/g_n(\tau_n)\ \phi(\tau_n)] = (n+1)[\ln\phi(\tau_{n+1}) - \ln\phi(\tau_n)] - (\tau_{n+1}-\tau_n)$$

$$= (n+1)\left[\rho\ \ln\frac{\tau_{n+1}}{\tau_n} + \int_{\tau_n}^{\tau_{n+1}} \frac{\varepsilon(y)}{y} dy\right] - \{(n+1)[\rho+\varepsilon(\tau_{n+1})] - n[\rho+\varepsilon(\tau_n)]\}\ .$$

To show $B_n \to 0$, we note that

$$\ln\phi(y) = \rho\ \ln y + \int_1^y \frac{\varepsilon(u)}{u}\, du\ ,\quad \frac{d}{dy}\ln\phi(y) = [\rho+\varepsilon(y)]/y$$

and

$$\frac{d^2}{dy^2}\ln\phi(y) = -\frac{1}{y}\{[\rho+\varepsilon(y)]/y - \varepsilon'(y)\}$$

so that

$$-\int_{\tau_n}^{\tau_{n+1}} y \frac{d^2}{dy^2} \ln \phi(y)dy = \int_{\tau_n}^{\tau_n} [\frac{\rho+\varepsilon(y)}{y} - \varepsilon'(y)]dy$$

$$= \rho \ln \frac{\tau_{n+1}}{\tau_n} + \int_{\tau_n}^{\tau_{n+1}} \frac{\varepsilon(y)}{y} dy - [\varepsilon(\tau_{n+1}) - \varepsilon(\tau_n)]$$

$$= (n+1)^{-1} B_n + (n+1)^{-1}[\rho+\varepsilon(\tau_n)] \ .$$

Hence,

$$B_n = -(n+1) \int_{\tau_n}^{\tau_{n+1}} y \frac{d^2}{dy^2} \ln \phi(y)dy - [\rho+\varepsilon(\tau_n)] \ .$$

However,

$$-\int_{\tau_n}^{\tau_{n+1}} y \frac{d^2}{dy^2} \ln \phi(y)dy < -\tau_{n+1} \int_{\tau_n}^{\tau_{n+1}} \frac{d^2}{dy^2} \ln \phi(y)dy$$

$$= -\tau_{n+1} [\frac{d}{dy} \ln \phi(y)]_{\tau_n}^{\tau_{n+1}} = \tau_{n+1}/n(n+1) \ .$$

Similarly,

$$-\int_{\tau_n}^{\tau_{n+1}} y \frac{d^2}{dy^2} \ln \phi(y)dy > \tau_n/n(n+1) \ .$$

Thus,

$$0 \leq \tau_n/n - [\rho+\varepsilon(\tau_n)] < B_n < \tau_{n+1}/n - [\rho+\varepsilon(\tau_n)] \to 0 \ .$$

This completes the proof of (2.11) and hence of Theorem 1.

4. Proof of Theorem 2.

We have $-\ln[1-F(x)] = \lambda^{-1}(\ln x)$, so that $F(x) = 1 - e^{-\lambda^{-1}(\ln x)}$

and

$$
(4.1)\qquad
\begin{aligned}
\mu_n' &= n \int_1^\infty x^{n-1}\, e^{-\lambda^{-1}(\ln x)}dx = n \int_0^\infty e^{ny-\lambda^{-1}(y)}dy \\
&= n \int_0^\infty \lambda'(y)\, e^{n\lambda(y)-y}dy = \int_0^\infty e^{n\lambda(y)-y}dy ,
\end{aligned}
$$

the last equality being obtained through an integration by parts. Let

$$
(4.2)\qquad \ln g_n(y) = n\lambda(y) - y = ny^\rho \exp\left\{\int_1^y \frac{\varepsilon(u)}{u}\, du\right\} - y .
$$

Then

$$
(4.3)\qquad \frac{d}{dy} \ln g_n(y) = n\lambda'(y) - 1 = n[\rho+\varepsilon(y)]\ \lambda(y)/y - 1
$$

and

$$
(4.4)\qquad
\begin{aligned}
\frac{d^2}{dy^2} \ln g_n(y) &= n\lambda''(y) = \frac{n\lambda(y)}{y^2}\{y\ \varepsilon'(y) - [\rho+\varepsilon(y)][1-\rho-\varepsilon(y)]\} \\
&= -n\lambda(y)[\rho(1-\rho) + b(y)]/y^2
\end{aligned}
$$

where

$$
(4.5)\qquad b(y) = \varepsilon(y)[1 - 2\rho - \varepsilon(y) - u(y)] .
$$

($u(y)$ is given by (3.12)). Since ε is slowly varying, and $\varepsilon(y) \to 0$, it follows from (2.8) that $b(y) \to 0$ as $y \to \infty$. If $\rho = 0$, then $\varepsilon(y) > 0$ and $b(y) > 0$. If $0 < \rho < 1$, then for y sufficiently large $b(y) < -\rho(1-\rho)$. If $\rho = 1$, $\varepsilon(y) < 0$ and $b(y) < 0$. Thus, $d^2/dy^2 \ln g_n(y)$ is negative for all y sufficiently large for $0 \le \rho \le 1$. Hence, $d/dy \ln g_n(y)$ decreases monotonically (to zero) as $y \to \infty$ and so there exists a unique positive real number τ_n at which (4.3) vanishes and $\ln g_n(y)$ attains a maximum at $y = \tau_n$. Clearly, τ_n satisfies $\tau_n = n[\rho+\varepsilon(\tau_n)]\lambda(\tau_n)$.

Since $d/dy \ln g_n(y)$ decreases monotonically, it follows that $\{\tau_n\}$ increases monotonically (for if τ_{n+1} were less than τ_n, we'd have $n^{-1} = [\rho+\varepsilon(\tau_n)]\lambda(\tau_n)/\tau_n < [\rho+\varepsilon(\tau_{n+1})]\lambda(\tau_{n+1})/\tau_{n+1} = (n+1)^{-1}$). It is easy to see that $\tau_n \to \infty$. The asymptotic behavior of $\{\tau_n\}$ and $\max_y \ln g_n(y) = n\,\lambda(\tau_n) - \tau_n$ depends on ρ, as follows:

(4.6) If $\rho = 0$, $\tau_n = n\varepsilon(\tau_n)\lambda(\tau_n)$; $\tau_n^{-1}\,\lambda(\tau_n) = L(\tau_n)\tau_n^{-1} \to 0$ (using 2.2);

and $\tau_n^{-1}[n\,\lambda(\tau_n) - \tau_n] = [1 - \varepsilon(\tau_n)]/\varepsilon(\tau_n) \to +\infty$.

(4.7) If $0 < \rho < 1$, $\tau_n = n[\rho+\varepsilon(\tau_n)]\lambda(\tau_n)$; $\tau_n^{-1}\,\lambda(\tau_n) = n^{-1}[\rho+\varepsilon(\tau_n)] \to 0$;

and $\tau_n^{-1}[n\lambda(\tau_n) - \tau_n] = [1-\rho-\varepsilon(\tau_n)]/[\rho+\varepsilon(\tau_n)] \to (1-\rho)/\rho$.

(4.8) If $\rho = 1$, $\tau_n = [1+\varepsilon(\tau_n)]\lambda(\tau_n)$; $\tau_n^{-1}\,\lambda(\tau_n) \to 0$; and

$\tau_n^{-1}[n\,\lambda(\tau_n) - \tau_n] = -\varepsilon(\tau_n)/[1+\varepsilon(\tau_n)] \to 0$. However,

$n\,\lambda(\tau_n) - \tau_n = -\varepsilon(\tau_n)\tau_n/[1+\varepsilon(\tau_n)] \to \infty$, using (2.2), recalling that $\varepsilon(x)$ is slowly varying.

As in the proof of Theorem 1, let

$$\mu_n' = \int_0^\infty g_n(y)dy = \tau_n \int_0^\infty g_n(\tau_n y)dy = \sum_{j=1}^{4} I_{nj} \ , \tag{4.9}$$

but with $I_{n3} = \tau_n \int_{1+\delta}^{c_n} g_n(\tau_n y)dy$ and $I_{n4} = \tau_n \int_{c_n}^{\infty} g_n(\tau_n y)dy$; c_n will be specified later. Let

$$h_n(x) = \ln g_n(\tau_n x) = n\,\lambda(\tau_n x) - \tau_n x \ , \tag{4.10}$$

then

$$\begin{cases} h_n'(x) = n \frac{d}{dx} \lambda(\tau_n x) - \tau_n = n[\rho+\varepsilon(\tau_n)]\lambda(\tau_n x)/x - \tau_n \\ \\ h_n''(x) = -n\,\lambda(\tau_n x)\{\rho(1-\rho) + b(\tau_n x)\}/x^2 \end{cases} \tag{4.11}$$

(noting 4.5). Note that $h_n(1) = \ln g_n(\tau_n)$ and $h_n'(1) = 0$. We will now show that

$$\begin{aligned} &\frac{I_{n2}}{\sqrt{2\pi\,\tau_n/(1-\rho)}\; g_n(\tau_n)} \to 1 \qquad \text{if} \quad 0 \le \rho < 1 \\ &\frac{I_{n2}}{\sqrt{-2\pi\,\tau_n/\varepsilon(\tau_n)}\; g_n(\tau_n)} \to 1 \qquad \text{if} \quad \rho = 1 \ . \end{aligned} \tag{4.12}$$

From (4.11), we see that

$$I_{n2} = \tau_n \int_{1-\delta}^{1+\delta} g_n(\tau_n x) = \tau_n\, g_n(\tau_n) \int_{1-\delta}^{1+\delta} \exp\{h_n''(\theta_n)(x-1)^2/2\}dx \tag{4.13}$$

with $1-\delta < \theta_n < 1+\delta$. To prove (4.12), we first show that

$$(4.14) \qquad (1+\delta)^{-2} \le \liminf \frac{[-h_n''(\theta_n)]}{\tau_n} \le \limsup \frac{[-h_n''(\theta_n)]}{\tau_n} \le (1-\delta)^{-2} .$$

If $\rho = 0$, from (4.5) and (4.11), we get

$$\begin{aligned}
-h_n''(\theta_n) &= n\ \lambda(\tau_n\ \theta_n)\theta_n^{-2}\ \varepsilon(\tau_n\ \theta_n)[1-\varepsilon(\tau_n\ \theta_n) - u(\tau_n\theta)] \\
&< \frac{n\ \lambda[(1+\delta)\tau_n]}{(1-\delta)^2}\ \varepsilon[(1-\delta)\tau_n][1-\varepsilon(\tau_n\ \theta_n) - u(\tau_n\theta)] \\
&= \frac{\tau_n}{(1-\delta)^2}\ \frac{\lambda[(1+\delta)\tau_n]}{\lambda(\tau_n)}\ [1-\varepsilon(\tau_n\ \theta_n) - u(\tau_n\theta)]
\end{aligned}$$

so that

$$\limsup[-h_n''(\theta_n)/\tau_n] \le (1-\delta)^{-2} , \quad \text{using (2.1) and (2.8).}$$

Similarly, $\liminf[-h_n''(\theta_n)/\tau_n] \ge (1+\delta)^{-2}$.

If $0 < \rho < 1$, then

$$\begin{aligned}
-h_n''(\theta_n) &\le n\ \lambda[(1+\delta)\tau_n]\{\rho(1-\rho) + b(\tau_n\ \theta_n)\}/(1-\delta)^2 \\
&= \frac{n\ \lambda(\tau_n)}{(1-\delta)^2}\ \frac{\lambda[(1+\delta)\tau_n]}{\lambda(\tau_n)}\{\rho(1-\rho) + b(\tau_n\ \theta_n)\} \\
&= \frac{\tau_n}{[\rho+\varepsilon(\tau_n)](1-\delta)^2}\ \frac{\lambda[(1+\delta)\tau_n]}{\lambda(\tau_n)}\{\rho(1-\rho) + b(\tau_n\ \theta_n)\}
\end{aligned}$$

so that

$$\lim \sup[-h_n''(\theta_n)/\tau_n] \leq (1-\rho)(1-\delta)^{-2} .$$

Similarly, $\lim \inf[-h_n''(\theta_n)/\tau_n] \geq (1-\rho)(1+\delta)^{-2}$.

If $\rho = 1$, then from (4.5) and (4.11), and proceeding as above,

$$-h_n''(\theta_n) \leq - \frac{n\ \lambda[(1+\delta)\tau_n]\varepsilon[(1-\delta)\tau_n]}{(1-\delta)^2} [1+\varepsilon(\tau_n\ \theta_n)+u(\tau_n\theta)]$$

$$= \frac{\tau_n\ \varepsilon(\tau_n)}{(1-\delta)^2[1+\varepsilon(\tau_n)]} \frac{\lambda[(1+\delta)\tau_n]\varepsilon[(1-\delta)\tau_n]}{\lambda(\tau_n)\ \varepsilon(\tau_n)}[1+\varepsilon(\tau_n\ \theta_n)+u(\tau_n\theta)]$$

so that

$$\lim \sup[-h_n''(\theta_n)/-\tau_n\ \varepsilon(\tau_n)] \leq (1-\delta)^{-2} .$$

Similarly, $\lim \inf[-h_n''(\theta_n)/-\tau_n\ \varepsilon(\tau_n)] \geq (1+\delta)^{-2}$. This establishes (4.14) for all $\rho\ \varepsilon[0,1]$. (4.12) now follows easily from (4.13) and (4.14), noting that $h_n''(\theta_n) \to -\infty$ uniformly for $\theta_n\ \varepsilon[1-\delta,\ 1+\delta]$ as $n \to \infty$ or $h_n''(\theta_n)/h_n''(1) \to 1$ uniformly as $n \to \infty$. If $0 \leq \rho < 1$, $h_n''(1) \simeq -(1-\rho)\tau_n$ while if $\rho = 1$ we have $h_n''(1) \simeq \tau_n\ \varepsilon(\tau_n)$. But, also, $h_n''(1) = -n\lambda''(\tau_n)$, so that (4.12) can alternatively be asserted as

$$\lim_{n\to\infty} I_{n2}/\alpha_n = 1$$

where

$$(4.15)\quad \alpha_n = \tau_n\sqrt{-2\pi/n\lambda''(\tau_n)}\ g_n(\tau_n), \quad \text{with}\quad n\ \lambda''(\tau_n) = \left\{\begin{matrix} (1-\rho)\tau_n\ , & 0 \leq \rho \leq 1 \\ \tau_n\ \varepsilon(\tau_n), & \rho = 1 \end{matrix}\right\} .$$

Next we show that $I_{nj}/\alpha_n \to 0$, for $j = 1, 3, 4$. We have

$$I_{n1}/\alpha_n < (1-\delta)[-n\ \lambda''(\tau_n)]^{1/2}\ g_n[(1-\delta)\tau_n]/\sqrt{2\pi}\ g_n(\tau_n)$$

$$= \exp(-n)\{\lambda(\tau_n) - \lambda[(1-\delta)\tau_n] - \delta\ \tau_n/n - \ln[-n\ \lambda''(\tau_n)]/2n\}\ ,$$

with $c = (1-\delta)/\sqrt{2\pi}$. Let a_n denote the quantity in brackets. Then

$$na_n = n\ \lambda(\tau_n)\{1 - \gamma_n(\delta) - \delta\tau_n/[n\ \lambda(\tau_n)] - \ln[-n\ \lambda''(\tau_n)]/2n\ \lambda(\tau_n)\}$$

with

$$\text{(4.16)} \qquad \gamma_n(\delta) = \lambda[(1-\delta)\tau_n]/\lambda(\tau_n) = (1-\delta)^{\rho} \exp\left\{- \int_{(1-\delta)\tau_n}^{\tau_n} \frac{\varepsilon(u)}{u}\, du\right\} .$$

If $\rho = 0$, then

$$na_n = \frac{\tau_n}{\varepsilon(\tau_n)} [1 - \gamma_n(\delta) - \delta\ \varepsilon(\tau_n) - \varepsilon(\tau_n)\ln\ \tau_n/2\ \tau_n]$$

$$\geq \frac{\tau_n}{\varepsilon(\tau_n)} [1 - \exp\{\varepsilon(\tau_n)\ln(1-\delta)\} - \delta\ \varepsilon(\tau_n) - \varepsilon(\tau_n)\ln\tau_n/2\tau_n]$$

$$\geq \frac{1}{2}\ \tau_n[[1 - \varepsilon(\tau_n)]\delta^2 - \ln\ \tau_n/\tau_n] \to \infty\ ,$$

using (i) the fact that $\gamma_n(\delta) \leq \exp[\varepsilon(\tau_n)\ln(1-\delta)]$, which is valid since $\varepsilon(x) \downarrow 0$ in this case, and (ii) the inequality

$$\text{(4.17)} \qquad 1 - (1-x)^a - ax \geq a(1-a)x^2/2\ .$$

If $0 < \rho < 1$, then

$$na_n = \frac{\tau_n}{[\rho+\varepsilon(\tau_n)]}\{1 - \gamma_n(\delta) - [\rho+\varepsilon(\tau_n)]\delta - [\rho+\varepsilon(\tau_n)]\ln[(1-\rho)\tau_n]/2\tau_n\}$$

$$\geq \frac{\tau_n}{[\rho+\varepsilon(\tau_n)]}\{1 - (1-\delta)^{\rho-|\beta_n|} - [\rho+\varepsilon(\tau_n)]\delta - [\rho+\varepsilon(\tau_n)]\ln[(1-\rho)\tau_n]/2\tau_n\}$$

$$\geq \frac{\tau_n}{2[\rho+\varepsilon(\tau_n)]}\{\rho(1-\rho)\delta^2 - [\rho+\varepsilon(\tau_n)]\ln[(1-\rho)\tau_n]/\tau_n + V_n\} \to \infty$$

using (4.17) again and where

$$V_n = -[\beta_n+\varepsilon(\tau_n)]\delta + [(2\rho-1)+\beta_n]\beta_n\ \delta^2/2\ , \quad \beta_n = \varepsilon[(1-\rho)\tau_n]\ .$$

Finally, if $\rho = 1$,

$$na_n = \frac{\tau_n}{[1+\varepsilon(\tau_n)]}\{1 - \gamma_n(\delta) - [1+\varepsilon(\tau_n)]\delta - [1+\varepsilon(\tau_n)]\ln[-\tau_n\ \varepsilon(\tau_n)]/2\tau_n\}$$

$$\geq \frac{\tau_n}{[1+\varepsilon(\tau_n)]}\{1 - (1-\delta)^{1+\varepsilon(\tau_n)} - [1+\varepsilon(\tau_n)]\delta - [1+\varepsilon(\tau_n)]\ln[-n\tau_n\ \varepsilon(\tau_n)]/2\tau_n\}$$

$$\geq \frac{\tau_n}{2}\ \{-\varepsilon(\tau_n)\delta^2 - [1+\varepsilon(\tau_n)]\ln[-\tau_n\ \varepsilon(\tau_n)]/\tau_n\}$$

$$= -\frac{\tau_n\ \varepsilon(\tau_n)}{2}\ \{\delta^2 - [1+\varepsilon(\tau_n)]\ln[-\tau_n\ \varepsilon(\tau_n)]/[-\tau_n\ \varepsilon(\tau_n)] \to \infty\ ,$$

on noting that in this case (i.e., $\rho = 1$), $\varepsilon(x) \uparrow 0$ and recalling (4.8).

This completes showing that $I_{n1}/\alpha_n \to 0$, for $0 \leq \rho \leq 1$.

We now show that $I_{n4}/\alpha_n \to 0$. From (4.9), we have

$$I_{n4}/\alpha_n = [\tau_n/\alpha_n] \int_{c_n}^{\infty} g_n(\tau_n x)dx$$

$$= \sqrt{-n\ \lambda''(\tau_n)/2\pi}\ e^{\tau_n} \int_{c_n}^{\infty} \exp[n\{\lambda(\tau_n x) - \lambda(\tau_n)\} - \tau_n x]dx\ .$$

Write

$$n\{\lambda(\tau_n x) - \lambda(\tau_n)\} - \tau_n x = -\tau_n\ x\{1 - n[\lambda(\tau_n x) - \lambda(\tau_n)]/\tau_n x\}$$

$$= -\tau_n x\ k_n(x)$$

where

$$1 - k_n(x) = n[\lambda(\tau_n x) - \lambda(\tau_n)]/\tau_n x$$

$$= \frac{n\ \lambda(\tau_n)}{\tau_n x}\left[\frac{\lambda(\tau_n x)}{\lambda(\tau_n)} - 1\right] = \left[x^{\rho}\ \exp\left\{\int_{\tau_n}^{\tau_n x} \frac{\varepsilon(u)}{u}\ du\right\} - 1\right]\Big/[\rho+\varepsilon(\tau_n)]x$$

We shall bound $k_n(x)$ from below.

If $\rho = 0$, then $\varepsilon(u) \downarrow 0$, so that

$$1 - k_n(x) < [e^{\varepsilon(\tau_n)\ln x} - 1]/x\ \varepsilon(\tau_n) = [x^{\varepsilon(\tau_n)} - 1]/x\ \varepsilon(\tau_n) \equiv f_n(x)\ .$$

The maximum value of $f_n(x)$ occurs at $x_n' = [1-\varepsilon(\tau_n)]^{1/\varepsilon(\tau_n)}$, at which $f_n(x_n') = [1-\varepsilon(\tau_n)]^{1/\varepsilon(\tau_n)}/[1-\varepsilon(\tau_n)] \to e^{-1}$. Thus, for any $\eta > 0$ there exists a N such that for all $n > N$, $1 - k_n(x) < e^{-1} + \eta$, i.e., $k_n(x) > 1 - e^{-1} - \eta$, so that if we take $\eta = 1/2$, say, then

$k_n(x) > \frac{1}{2} - e^{-1} = k > 0$, at least for $n > N$. Then

$$I_{n4}/\alpha_n \le \sqrt{\frac{\tau_n}{2\pi}}\, e^{\tau_n} \int_{c_n}^{\infty} e^{-\tau_n xk}\, dx$$

$$= e^{-\tau_n (kc_n - 1)} \Big/ \sqrt{\frac{\tau_n}{2\pi}} \to 0$$

if we choose c_n such that $k\, c_n - 1 > 0$, i.e., if $c_n > 1/k$. We take $c_n = 2/k$.

If $0 < \rho < 1$, then

$$1 - k_n(x) = \left[\frac{\lambda(\tau_n x)}{\lambda(\tau_n)} - 1\right] \Big/ [\rho + \varepsilon(\tau_n)]x \to (x^{\rho} - 1)/\rho x = f(x) ,$$

since λ varies regularly with exponent ρ. The maximum value of $f(x)$ occurs at $x_0 = 1/[1-\rho]^{1/\rho}$, at which $f(x_0) = (1-\rho)^{1/\rho}/(1-\rho) < 1$ for $0 < \rho < 1$. Thus, for all $x > x_0$ and n sufficiently large, $1 - k_n(x) < f(x_0) + \eta$, or $k_n(x) > 1 - f(x_0) - \eta > \theta > 0$ if we take $\eta < 1 - (1-\rho)^{1/\rho}/(1-\rho) - \theta$. Then

$$I_{n4}/\alpha_n \le \sqrt{\frac{(1-\rho)\tau_n}{2\pi}}\, e^{\tau_n} \int_{c_n}^{\infty} e^{-\tau_n x}\, dx$$

$$= \frac{1}{\theta} \sqrt{\frac{(1-\rho)}{2\pi\, \tau_n}}\, e^{-\tau_n (1-\theta c_n)} \to 0$$

if we choose $1 - \theta\, c_n > 0$, or $c_n > 1/\theta$. Take $c_n = 2/\theta$; we can let $\theta = [1-f(x_0)]/2$.

Finally, if $\rho = 1$, let x_n be such that $\ln L(x_n) = \int_1^{x_n} \frac{\varepsilon(y)}{y}\, dy =$ $= \ln[(1-\delta)/n]$; this is possible since $\lambda(x) = x\, L(x) \to +\infty$ with $L(x) \to 0$ as $x \to \infty$. We prove the following lemma about $\{x_n\}$.

<u>Lemma 4.1</u>. $\{x_n\} \uparrow +\infty$ and $\{x_n/\tau_n\} \to +\infty$.

<u>Proof</u>. We have

$$\tau_n = n[1+\varepsilon(\tau_n)]\lambda(\tau_n) = n[1+\varepsilon(\tau_n)]\tau_n \exp\left\{\int_1^{\tau_n} \frac{\varepsilon(y)}{y}\, dy\right\}$$

which implies that

$$\int_1^{\tau_n} \frac{\varepsilon(y)}{y}\, dy = -\ln\{n[1+\varepsilon(\tau_n)]\} .$$

Thus,

$$\int_{\tau_n}^{x_n} \frac{\varepsilon(y)}{y}\, dy = \int_1^{x_n} \frac{\varepsilon(y)}{y}\, dy - \int_1^{\tau_n} \frac{\varepsilon(y)}{y}\, dy = \ln(1-\delta) + \ln\{[1+\varepsilon(\tau_n)]\} < 0$$

which implies $x_n > \tau_n$ since $\varepsilon(y) < 0$. Clearly, $\int_{\tau_n}^{x_n} \frac{\varepsilon(y)}{y}\, dy \to \ln(1-\delta)$ as $n \to \infty$. But

$$\varepsilon(\tau_n)\ln \frac{x_n}{\tau_n} \le \int_{\tau_n}^{x_n} \frac{\varepsilon(y)}{y}\, dy \le \varepsilon(x_n)\ln \frac{x_n}{\tau_n} .$$

Let $0 < \gamma < -\ln(1-\delta)$, so that $\ln(1-\delta) + \gamma < 0$. From above it follows that there exists N such that for all $n > N$,

$$\varepsilon(\tau_n)\ln \frac{x_n}{\tau_n} < \ln(1-\delta) + \gamma$$

i.e., $\ln \frac{x_n}{\tau_n} > \frac{\ln(1-\delta)+\gamma}{\varepsilon(\tau_n)} \to \infty$, since $\varepsilon(\tau_n) \uparrow 0$. Thus, $x_n/\tau_n \to \infty$.

We now show that $I_{n4}/\alpha_n \to 0$ if we choose $c_n = x_n/\tau_n$. We have

$$I_{n4}/\alpha_n = \frac{\tau_n}{\alpha_n} \int_{x_n/\tau_n}^{\infty} g_n(\tau_n x)dx$$

$$= \frac{\sqrt{-\tau_n \ \varepsilon(\tau_n)}}{\sqrt{2\pi} \ g_n(\tau_n)} \int_{x_n/\tau_n}^{\infty} g_n(\tau_n x)dx = \frac{[1+\varepsilon(\tau_n)]}{\sqrt{-2\pi \ \tau_n \ \varepsilon(\tau_n)}} \int_{x_n/\tau_n}^{\infty} g_n(\tau_n x)dx$$

$$= \frac{[1+\varepsilon(\tau_n)]}{\sqrt{-2\pi \ \tau_n \ \varepsilon(\tau_n)}} \frac{1}{\tau_n} \int_{X_n}^{\infty} e^{n\lambda(x)-x} dx$$

$$< \frac{[1+\varepsilon(\tau_n)]}{\tau_n\sqrt{-2\pi \ \tau_n \ \varepsilon(\tau_n)}} \int_{x_n}^{\infty} e^{-\delta x} dx = \frac{[1+\varepsilon(\tau_n)]}{\delta \ \tau_n\sqrt{-2\pi \ \tau_n \ \varepsilon(\tau_n)}} e^{-\delta x_n} \to 0 ,$$

using $g_n(\tau_n) = n \ \lambda(\tau_n) - \tau_n = -\tau_n \ \varepsilon(\tau_n)/[1+\varepsilon(\tau_n)]$ (see 4.8), and the fact that for all $x > x_n$,

$$n \ \lambda(x)-x = n \ x \exp\left\{\int_1^x \frac{\varepsilon(y)}{y} dy\right\} - x < n \ x \exp\left\{\int^{x_n} \frac{\varepsilon(y)}{y} dy\right\} = -\delta x .$$

This completes showing $I_{n4}/\alpha_n \to 0$ for $0 \leq \rho \leq 1$.

At this point, we summarize the choices of c_n in I_{n3} and I_{n4} :

$$c_n = \left\{ \begin{array}{ll} 2/k = 2/[.5-e^{-1}] & \text{if} \ \ \rho = 0 \\ 2/\theta = 4/[1-(1-\rho)^{\rho/1-\rho}] & \text{if} \ \ 0 < \rho < 1 \\ X_n/\tau_n & \text{if} \ \ \rho = 1 \end{array} \right\} .$$

We still need to show that $I_{n3}/\alpha_n \to 0$. If $\rho = 0$,

then $c_n = 2/k$ and

$$I_{n3}/\alpha_n = \frac{\tau_n}{\sqrt{2\pi}\,\tau_n\, g_n(\tau_n)} \int_{1+\delta}^{2/k} g_n(\tau_n x)dx = \sqrt{\frac{\tau_n}{2\pi}} \int_{1+\delta}^{2/k} e^{h_n''(\theta_n)(x-1)^2/2}\, dx$$

on paralleling the calculation of I_{n2} in (4.10), (4.11), and (4.13); now, $1 + \delta < \theta_n < 2/k$. As for $h_n''(\theta_n)$ we have

$$-h_n''(\theta_n) \geq k^2\, n\, \lambda[(1+\delta)\tau_n]\, \varepsilon[2\tau_n/k]\{S(\tau_n\, \theta_n) - \varepsilon(\tau_n\, \theta_n)\}/4$$

$$= \left[\frac{k^2\, \lambda(\tau_n)\, \varepsilon(\tau_n)}{4}\right] \frac{\lambda[(1+\delta)\tau_n]\, \varepsilon[2\tau_n/k]}{\lambda(\tau_n)\, \varepsilon(\tau_n)} [1+o(1)]$$

so that $\lim\inf\{-h_n''(\theta_n)/\tau_n\} \geq k^2/4$. Hence, for all n sufficiently large, $h_n''(\theta_n) \leq -k^2\, \tau_n/2$, so that

$$I_{n3}/\alpha_n \leq \sqrt{\frac{\tau_n}{2\pi}} \int_{1+\delta}^{2/k} e^{-k^2(x-1)^2/8}\, dx \to 0 .$$

A similar calculation shows that $I_{n3}/\alpha_n \to 0$ for $0 < \rho < 1$. If $\rho = 1$, we need to show that

$$I_{n3}/\alpha_n = \frac{\tau_n}{\alpha_n} \int_{1+\delta}^{x_n/\tau_n} g_n(\tau_n x)dx$$

$$= \sqrt{\frac{-\tau_n\, \varepsilon(\tau_n)}{2\pi}} \int_{1+\delta}^{x_n/\tau_n} e^{h_n''(\theta_n)(x-1)^2/2}\, dx$$

where now $1 + \delta < \theta_n < x_n/\tau_n$. We first show that $h_n''(\theta_n) \to -\infty$, uniformly for $\theta_n\, \varepsilon[1+\delta,\, x_n/\tau_n]$. We have

$$-h_n''(\theta_n) = -n\,\lambda(\tau_n\,\theta_n)[1+\varepsilon(\tau_n\,\theta_n)+u(\tau_n\,\theta_n)]/\theta_n^2 \; .$$

Clearly, $\omega_n = 1+\varepsilon(\tau_n\,\theta_n) + u(\tau_n\,\theta_n) \to 1$ for $\theta_n = 1+\delta$ and x_n/τ_n and is bounded above and below by its values at these two points. Thus,

$$\begin{aligned}
-h_n''(\theta_n) &\geq n\,\lambda[(1+\delta)\tau_n]\{-\varepsilon(x_n)\}[1+o(1)]/\tau_n^2/x_n^2 \\
&= [-x_n\,\varepsilon(x_n)](x_n/\tau_n)\,n(1+\delta)\,\exp\left\{\int_1^{(1+\delta)\tau_n} \frac{\varepsilon(y)}{y}\,dy\right\} \\
&\geq [-x_n\,\varepsilon(x_n)](1-\delta)^2 = z_n^2\,, \quad z_n = \sqrt{-x_n\,\varepsilon(X_n)}\,(1-\delta) \; .
\end{aligned}$$

Here, we used the fact that for all n sufficiently large, $x_n \geq (1+\delta)\tau_n$ (which follows from Lemma 4.1) and

$$n(1+\delta)\,\exp\left\{\int_1^{(1+\delta)\tau_n} \frac{\varepsilon(y)}{y}\,dy\right\} \geq n(1+\delta)\,\exp\left\{\int_1^{x_n} \frac{\varepsilon(y)}{y}\,dy\right\} = 1-\delta^2 \; .$$

Hence,

$$\begin{aligned}
I_{n3}/\alpha_n &\leq \sqrt{\frac{-\tau_n\,\varepsilon(\tau_n)}{2\pi}} \int_{1+\delta}^{x_n/\tau_n} e^{-z_n^2(x-1)^2/2}\,dx \\
&< \frac{1}{1-\delta}\sqrt{\frac{-\tau_n\,\varepsilon(\tau_n)}{-x_n\,\varepsilon(x_n)}} \int_{z_n-1}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du \to 0
\end{aligned}$$

since $-x\,\varepsilon(x)$ varies regularly and is easily shown to be monotonically increasing, so that $[-\tau_n\,\varepsilon(\tau_n)] \leq [-x_n\,\varepsilon(x_n)]$.

This completes the proof that $I_{n3}/\alpha_n \to 0$ for $0 \leq \rho \leq 1$ and hence of (2.13) and (2.14).

To prove (2.15) and (2.16), we proceed as follows. Suppose $\tau_{n+1}/\tau_n \to 1$ (we will show later that this is the case if $0 \leq \rho < 1$ and for a class of slowly varying functions L in $\lambda(x) = x\,L(x)$). If $0 \leq \rho < 1$, then from (2.13) we have

$$\ln[\mu'_{n+1}/\mu'_n] \simeq (n+1)\lambda(\tau_{n+1}) - n\,\lambda(\tau_n) - (\tau_{n+1} - \tau_n)$$

$$= n[\lambda(\tau_{n+1}) - \lambda(\tau_n)] - (\tau_{n+1} - \tau_n) + \lambda(\tau_{n+1})$$

$$= n[\lambda(\tau_{n+1}) - \lambda(\tau_n)] - \{n[\rho+\varepsilon(\tau_{n+1})]\lambda(\tau_{n+1})]$$

$$- n[\rho+\varepsilon(\tau_n)]\lambda(\tau_n)\} + [1-\rho-\varepsilon(\tau_{n+1})]\lambda(\tau_{n+1})$$

$$= n(1-\rho)\{\lambda(\tau_{n+1}) - \lambda(\tau_n)\} - n\{\varepsilon(\tau_{n+1})\lambda(\tau_{n+1}) - \varepsilon(\tau_n)\lambda(\tau_n)\}$$

$$+ [1-\rho-\varepsilon(\tau_{n+1})]\lambda(\tau_{n+1})$$

$$= -n\int_{\tau_n}^{\tau_{n+1}} x\,\lambda''(x)dx + [1-\rho-\varepsilon(\tau_{n+1})]\lambda(\tau_{n+1}) .$$

The latter equality is obtained as follows. We know from (4.3) that $x\,\lambda'(x) = [\rho+\varepsilon(x)]\lambda(x)$ and from (4.4) that

$$x\,\lambda''(x) = \{x\,\varepsilon'(x) - [\rho+\varepsilon(x)][1-\rho-\varepsilon(x)]\}\lambda(x)/x$$

$$= \varepsilon'(x)\,\lambda(x) - [1-\rho-\varepsilon(x)]\,\frac{[\rho+\varepsilon(x)]\lambda(x)}{x}$$

$$= \varepsilon'(x)\,\lambda(x) - [1-\rho-\varepsilon(x)]\lambda'(x)$$

$$= [\varepsilon'(x)\,\lambda(x) + \varepsilon(x)\,\lambda'(x)] - (1-\rho)\,\lambda'(x)$$

$$= \frac{d}{dx}\,[\varepsilon(x)\,\lambda(x)] - (1-\rho)\,\lambda'(x)\;.$$

The equality now follows on integrating $x\,\lambda''(x)$ from τ_n to τ_{n+1}. But

$$-n\int_{\tau_n}^{\tau_{n+1}} x\,\lambda''(x)dx < -n\,\tau_{n+1}\int_{\tau_n}^{\tau_{n+1}} \lambda''(x)dx = -n\,\tau_{n+1}\{\lambda'(\tau_{n+1})-\lambda'(\tau_n)\}$$

(4.18)

$$= -n\,\tau_{n+1}\{\frac{1}{n+1} - \frac{1}{n}\} = \tau_{n+1}/(n+1)\;.$$

Similarly,

$$-n\int_{\tau_n}^{\tau_{n+1}} \lambda''(x)dx > \tau_n/(n+1)\;. \tag{4.19}$$

Therefore,

$$\frac{\ln[\mu'_{n+1}/\mu'_n]}{\lambda(\tau_{n+1})} < \frac{\tau_{n+1}}{(n+1)\lambda(\tau_{n+1})} + 1-\rho-\varepsilon(\tau_{n+1}) = 1$$

since $\tau_{n+1}/[(n+1)\lambda(\tau_{n+1})] = \rho+\varepsilon(\tau_{n+1})$. Also,

$$\frac{\ln[\mu'_{n+1}/\mu'_n]}{\lambda(\tau_{n+1})} > \frac{\tau_n}{(n+1)\lambda(\tau_{n+1})} + 1-\rho-\varepsilon(\tau_{n+1})$$

$$= \frac{\tau_n[\rho+\varepsilon(\tau_{n+1})]}{\tau_{n+1}} + 1-\rho-\varepsilon(\tau_{n+1}) \to \rho+1-\rho = 1 .$$

Now suppose $\rho = 1$ and $\tau_{n+1}/\tau_n \to 1$. From (2.14), we have

$$\ln[\mu'_{n+1}/\mu'_n] \simeq (n+1)\lambda(\tau_{n+1}) - n\,\lambda(\tau_n) - (\tau_{n+1} - \tau_n) + \frac{1}{2}\ln c_n$$

$$= -n\int_{\tau_n}^{\tau_{n+1}} X\,\lambda''(x)dx - \varepsilon(\tau_{n+1})\lambda(\tau_{n+1}) + \frac{1}{2}\ln c_n$$

where $c_n = \tau_{n+1}\,\varepsilon(\tau_n)/\tau_n\,\varepsilon(\tau_{n+1})$. Since $\varepsilon(x)$ is a slowly varying function, it can be represented as $\varepsilon(x) = a(x)\exp\{\int_1^x \frac{\nu(y)}{y}dy\}$ where $a(x) \to a < \infty$ and $\nu(y) \to 0$ as $y \to \infty$. Therefore,

$$\ln\frac{\varepsilon(\tau_{n+1})}{\varepsilon(\tau_n)} = \ln\frac{a(\tau_{n+1})}{a(\tau_n)} + \int_{\tau_n}^{\tau_{n+1}} \frac{\nu(y)}{y}dy \to 0$$

since $a(\tau_{n+1})/a(\tau_n) \to 1$ and $\left|\int_{\tau_n}^{\tau_{n+1}} \frac{\nu(y)}{y}dy\right| \leq \int_{\tau_n}^{\tau_{n+1}} \left|\frac{\nu(y)}{y}\right| dy$ $\leq \max[\nu(\tau_n), \nu(\tau_{n+1})]\ln\frac{\tau_{n+1}}{\tau_n} \to 0$ if τ_{n+1}/τ_n converges to a finite limit. Thus, $\ln\varepsilon(\tau_n)/\varepsilon(\tau_{n+1}) \to 0$; clearly, $\ln\tau_{n+1}/\tau_n \to 0$. It follows from (4.18) and (4.19) that $\ln[\mu'_{n+1}/\mu'_n]/\lambda(\tau_{n+1}) \to 1$.

If $\rho = 1$ and $\tau_{n+1}/\tau_n \to L > 1$, then from (2.14) and (4.8),

$$\ln[\mu'_{n+1}/\mu'_n] \simeq (n+1)\ \lambda(\tau_{n+1}) - n\ \lambda(\tau_n) - (\tau_{n+1}-\tau_n) + \frac{1}{2}\ln c_n$$

$$= \frac{\tau_{n+1}}{1+\varepsilon(\tau_{n+1})} - \frac{\tau_n}{1+\varepsilon(\tau_n)} - (\tau_{n+1}-\tau_n) + \frac{1}{2}\ln c_n$$

$$= -\frac{\tau_{n+1}\ \varepsilon(\tau_{n+1})}{1+\varepsilon(\tau_{n+1})} + \frac{\tau_n\ \varepsilon(\tau_n)}{1+\varepsilon(\tau_n)} + \frac{1}{2}\ln c_n$$

so that

$$\frac{\ln[\mu'_{n+1}/\mu'_n]}{-\tau_{n+1}\ \varepsilon(\tau_{n+1})} \simeq \frac{1}{1+\varepsilon(\tau_{n+1})} - \frac{\tau_n\ \varepsilon(\tau_n)}{\tau_{n+1}\ \varepsilon(\tau_{n+1})[1+\varepsilon(\tau_n)]} - \frac{\ln c_n}{2\ \tau_{n+1}\ \varepsilon(\tau_{n+1})}\ .$$

Consider

$$0 < \frac{0 \le \ln c_n}{-\tau_{n+1}\ \varepsilon(\tau_{n+1})} = \ln\left(\frac{\tau_{n+1}\ \varepsilon(\tau_n)}{\tau_n\ \varepsilon(\tau_{n+1})}\right)\Big/ -\tau_{n+1}\ \varepsilon(\tau_{n+1})$$

$$= -\int_{\tau_n}^{\tau_{n+1}} \left(\frac{1}{y} - \frac{\varepsilon'(y)}{\varepsilon(y)}\right)dy \Big/ \int_1^{\tau_{n+1}} \frac{d}{dy}\,[y\ \varepsilon(y)]dy$$

$$= \int_{\tau_n}^{\tau_{n+1}} \frac{1}{y}\left[1 - \frac{y\varepsilon'(y)}{\varepsilon(y)}\right]dy \Big/ \int_1^{\tau_{n+1}} \{-\varepsilon(y)\}\left[1 + \frac{y\varepsilon'(y)}{\varepsilon(y)}\right] dy$$

$$\le \int_{\tau_n}^{\tau_{n+1}} \frac{[1-u(y)]}{y}\ dy \Big/ \int_{\tau_n}^{\tau_{n+1}} [-\varepsilon(y)][1+u(y)]dy$$

where $u(y) = y\ \varepsilon'(y)/\varepsilon(y) \to 0$ as $y \to \infty$; $u(y) < 0$. Thus, for any $\delta > 0$ there exists N such that $1 - u(y) < 1 + \delta$ and $1 + u(y) > 1 - \delta$. Also, $-y\ \varepsilon(y) \uparrow +\infty$, so that for $\tau_n < y < \tau_{n+1}$,
$-\tau_n\ \varepsilon(\tau_n) < -y\ \varepsilon(y) < -\tau_{n+1}\ \varepsilon(\tau_{n+1})$, so that
$-\tau_n\ \varepsilon(\tau_n)/y < -\varepsilon(y) < -\tau_{n+1}\ \varepsilon(\tau_{n+1})/y$. Therefore, for $n > N$,

$$\ln c_n/[-\tau_{n+1}\,\varepsilon(\tau_{n+1})] < (1+\delta)/[-\tau_n\,\varepsilon(\tau_n)](1-\delta) \to 0 \quad \text{as} \quad n \to \infty .$$

If $\lim \tau_{n+1}/\tau_n = L < \infty$, then it is easily shown that $\varepsilon(\tau_{n+1})/\varepsilon(\tau_n) \to 1$, so that $\tau_{n+1}\,\varepsilon(\tau_{n+1})/\tau_n\,\varepsilon(\tau_n) \to L$. In this case, $\ln[\mu'_{n+1}/\mu'_n]/[-\tau_{n+1}\,\varepsilon(\tau_{n+1})] \to 1 - L^{-1} = (L-1)/L$.

If $L = +\infty$, we claim that $\tau_{n+1}\,\varepsilon(\tau_{n+1})/\tau_n\,\varepsilon(\tau_n) \to \infty$ also. To see this, we have

$$\begin{aligned}\frac{1}{a_n}\,\frac{\lambda(\tau_{n+1})}{\lambda(\tau_n)}\,\frac{\varepsilon(\tau_{n+1})}{\varepsilon(\tau_n)} &= \frac{\tau_{n+1}\,\varepsilon(\tau_{n+1})}{\tau_n\,\varepsilon(\tau_n)}\exp\left\{\int_{\tau_n}^{\tau_{n+1}}[\varepsilon(y)/y]dy\right\}\\ &= \exp\left\{\int_{\tau_n}^{\tau_{n+1}}\left[\frac{1}{y}+\frac{\varepsilon'(y)}{\varepsilon(y)}+\frac{\varepsilon(y)}{y}\right]dy\right\}\\ &= \exp\left\{\int_{\tau_n}^{\tau_{n+1}}y^{-1}\left[1+\varepsilon(y)+\frac{y\,\varepsilon'(y)}{\varepsilon(y)}\right]dy\right\}\end{aligned}$$

where $a_n = (n+1)[1+\varepsilon(\tau_{n+1})]/n[1+\varepsilon(\tau_n)] \to 1$. Both $\varepsilon(y)$ and $y\,\varepsilon'(y)/\varepsilon(y)$ are negative and converge to zero as $y \to \infty$, so that for any $\delta > 0$ there exists N such that

$$1+\varepsilon(y) + y\,\varepsilon'(y)/\varepsilon(y) > 1-\delta \quad \text{for all} \quad n > N, \text{ for } \tau_n < y < \tau_{n+1} .$$

Thus,

$$\frac{\lambda(\tau_{n+1})\,\varepsilon(\tau_{n+1})}{\lambda(\tau_n)\,\varepsilon(\tau_n)} \geq a_n \exp\{(1-\delta)\ln[\tau_{n+1}/\tau_n]\} = a_n(\tau_{n+1}/\tau_n)^{1-\delta} \to +\infty .$$

But

$$\frac{\lambda(\tau_{n+1})}{\lambda(\tau_n)} \frac{\tau_n}{\tau_{n+1}} = a_n \exp\left\{\int_{\tau_n}^{\tau_{n+1}} \frac{\varepsilon(y)}{y} dy\right\} \to 1 \quad \text{as} \quad n \to \infty ,$$

so that $\lambda(\tau_{n+1})/\lambda(\tau_n) \sim \tau_{n+1}/\tau_n$. This proves the claim above. Thus, if $\tau_{n+1}/\tau_n \to \infty$,

$$\ln[\mu'_{n+1}/\mu'_n]/[-\tau_n \, \varepsilon(\tau_{n+1})] \to 1 .$$

This completes the proof of (2.15) and (2.16) and hence of Theorem 2.

5. Examples.

To illustrate the application of the results here, we now present several examples.

A. Slowly Varying Functions.

We let $\phi(x) = \exp\{\int_1^x \frac{\varepsilon(y)}{y} dy\}$ where $\varepsilon(y) \downarrow 0$ as $y \to \infty$. As examples we have

1. $\phi(x) = (\ln x)^\beta$, $\varepsilon(x) = \beta/\ln x$ for $x \geq e$ and 0 for $x \leq e$.

2. $\phi(x) = (\ln \ln x)^\beta \equiv (\ell_2 \, x)^\beta$, $\varepsilon(x) = \beta[(\ln x)(\ln\ln x)]^{-1}$, $x \geq e^2$ and 0 for $x \leq e^2$.

3. $\phi(x) = (\ell_k \, x)^\beta$, $\varepsilon(x) = \beta/[(\ln x)(\ln \ln x)\ldots(\ln \ln \ldots \ln x)]$
$\equiv \beta/[\ell_1 \, x)(\ell_2 \, x)\ldots(\ell_k \, x)]$, for $x \geq e(k)$

4. $\phi(x) = \exp\{(\ln x)^\beta\}$, $\varepsilon(x) = \beta/(\ln x)^{1-\beta}$, for $x \geq e$ and $0 < \beta < 1$.

If $\phi(x)$ is a slowly varying function, then Theorem 1 applies; we have $\tau_n = n[\rho + \varepsilon(\tau_n)] = n \, \varepsilon(\tau_n)$,

$$\mu_n' \simeq \sqrt{2\pi\,\tau_n}\; g_n(\tau_n) = \sqrt{2\pi\,\tau_n}\;[\phi(\tau_n)]^n\; e^{-\tau_n}$$

and

$$\mu_{n+1}'/\mu_n' \simeq \phi(\tau_n) \;,$$

from (2.9)-(2.11).

To use Theorem 1, we need to approximate τ_n ; the following lemma is useful in this regard.

Lemma 5.1. Let $\phi(x) = \exp\{\int_1^x \frac{\varepsilon(y)}{y}\}$, with $\varepsilon(y) \downarrow 0$ as $y \to \infty$ and $\tau_n = n\,\varepsilon(\tau_n)$. Let $\tau_{n1} = n$, $\tau_{n2} = n\,\varepsilon(\tau_{n1})$, ..., $\tau_{nk} = n\,\varepsilon(\tau_{n,k-1})$. Then for all $k \geq 1$, $\tau_{n,2k} \leq \tau_n \leq \tau_{n,2k-1}$.

Proof. Let $f(x) = n\,\varepsilon(x)/x$. Then $f(n) = n\,\varepsilon(n)/n = \varepsilon(n) < 1$, so that $\tau_n < n = \tau_{n1}$, $\tau_{n2} = n\,\varepsilon(n) < n = \tau_{n1}$, so that

$$f(\tau_{n2}) = n\,\varepsilon(\tau_{n2})/\tau_{n2} = \varepsilon(\tau_{n2})/\varepsilon(\tau_{n1}) > 1$$

and hence $\tau_n > \tau_{n2}$. Continuing, $\tau_{n3} = n\,\varepsilon(\tau_{n2}) > n\,\varepsilon(\tau_{n1}) = \tau_{n2}$ so that

$$f(\tau_{n3}) = n\,\varepsilon(\tau_{n3})/\tau_{n3} = n\,\varepsilon(\tau_{n3})/n\,\varepsilon(\tau_{n2}) < 1$$

and hence $\tau_n < \tau_{n3}$. Continuing in this manner, it can be shown that (a) $\tau_{n1} > \tau_{n3} > \ldots > \tau_{n,2k+1}$ ... , (b) $\tau_{n2} < \tau_{n4} < \ldots < \tau_{n,2k}$, and (c) for any k , $\tau_{n,2k} < \tau_{n,2k+1}$. Also, by induction, we can show

that $f(\tau_{n,2k+1}) < 1$, $f(\tau_{n,2k}) > 1$. The result follows from these facts.

Example 1. $\phi(x) = \ln x$, so that $\phi^{-1}(y) = e^y$ and $1-F(y) = e^{-e^y}$; also $\varepsilon(y) = 1/\ln y$ and τ_n satisfies $\tau_n = n\,\varepsilon(\tau_n) = n/\ln \tau_n$. By Lemma (5.1), we have $\tau_{n1} = n$, $\tau_{n2} = n/\ln n$, $\tau_{n3} = n/[\ln \frac{n}{\ln n}] =$ $= n/[\ln n - \ln \ln n]$, $\tau_{n4} = n/[\ell_1 n - \ell_2\, n + \ell_3\, n]$, ..., $\tau_{nk} = n / [\Sigma_{j=1}^{k-1} (-1)^j\, \ell_j\, n]$. In particular, we have

$$\tau_{n2} = n/\ln n < \tau_n < n/[\ln n - \ln \ln n] = \tau_{n3}$$

and

$$1 \le \tau_n/[n/\ln n] \le \frac{n/[\ln n - \ln \ln n]}{[n/\ln n]} = \ln n/[\ln n - \ln \ln n] \to 1$$

so that $\tau_n \sim n/\ln n$. Thus,

$$\mu'_n \simeq \sqrt{2\pi\, n/\ln n}\, \{\ln[n/\ln n]\}^n\, e^{-n/\ln n}$$

and

$$\mu'_{n+1}/\mu'_n \simeq \phi(\tau_n) = \ln[n/\ln n] = \ln n - \ln \ln n \simeq \ln n \text{ .}$$

Example 2. $\phi(x) = \ln \ln x$, $\phi^{-1}(y) = e^{e^y}$, $\varepsilon(y) = [(\ln y)(\ln \ln y)]^{-1}$, for $y \ge e^2$; τ_n satisfies $\tau_n = n\,\varepsilon(\tau_n) = n/[(\ln \tau_n)(\ln \ln \tau_n)]$. By Lemma 5.1, we find that

$$n/[(\ln n)(\ln\ln n)] < \tau_n < n/[\ln n - \ell_2 n - \ell_3 n][\ln\{\ln n - \ell_2 n - \ell_3 n\}]$$

so that $\tau_n \sim n/[(\ln n)(\ln\ln n)]$. Thus,

$$\mu_n' \simeq \sqrt{2\pi\, n/[(\ln n)(\ln\ln n)]}\ \{\ln n/[(\ln n)(\ln\ln n)]\}^n\, e^{-n/[\ln n][\ln\ln n]}$$

and

$$\mu_{n+1}'/\mu_n' \simeq \ln\ln\{n/[\ln n][\ln\ln n]\} \simeq \ln\ln n .$$

<u>Example 3</u>. $\phi(x) = \ell_k(x) = \ln\ln \ldots \ln x$, for $x \geq e_k$ and 0 for $x < e_k$. $\varepsilon(y) = [(\ell_1 y)(\ell_2 y)\ldots(\ell_k y)]^{-1}$ and $\tau_n = n\,\varepsilon(\tau_n)$; by Lemma 5.1 we find that τ_n lies between τ_{n1} and τ_{n2} with $\tau_{n1} = n\,\varepsilon(n)$ and $\tau_{n2} = n\,\varepsilon(\tau_{n1})$ with $\tau_n \simeq \tau_{n1}$.

As a generalization of Examples 1 and 2 above, we find

$$\mu_{n+1}'/\mu_n' \simeq \ell_k\, n = \ln\ln \ldots \ln n .$$

It is easily seen that $\phi^{-1}(y) = e_k(y)$ where $e_1(y) = e^y$, $e_2(y) = e^{e^y}$, and, in general, $e_k(y) = e_{k-1}(e^y)$.

<u>Example 4</u>. $\phi(x) = e^{(\ln x)^\beta}$, $0 < \beta < 1$; $\varepsilon(y) = \beta/(\ln x)^{1-\beta}$, $\tau_{n1} = n\,\varepsilon(n) = n\,\beta/(\ln n)^{1-\beta}$, $\tau_{n2} = n\,\varepsilon(\tau_{n1}) = n\,\beta/[\ln n\beta - (1-\beta)\ln\ln n]^{1-\beta}$ and $\tau_{n1} < \tau_n < \tau_{n2}$; $\tau_n/\tau_{n1} \to 1$, so $\tau_n \sim \tau_{n1}$. Also, $\phi^{-1}(y) = e^{(\ln y)^{1/\beta}}$, so $1-F(y) = e^{-e^{(\ln y)^{1/\beta}}}$. In this case, $\mu_{n+1}'/\mu_n' \simeq \exp\{(\ln n)^\beta\}$.

B. Regularly Varying Functions.

We let $\phi(x) = x^{\rho} \exp\{\int_1^x \frac{\varepsilon(y)}{y} dy\}$, with $\rho > 0$; $\varepsilon(y) \to 0$ as $y \to \infty$. τ_n satisfies $\tau_n = n[\rho+\varepsilon(y)]$ and $\tau_n/n \to \rho$; $\mu_n' \simeq \sqrt{2\pi \tau_n}\, \phi(\tau_n)\, e^{-\tau_n}$ and $\mu_{n+1}'/\mu_n' \simeq \phi(\tau_n)$.

<u>Example 5</u>. $\phi(x) = x^{\rho}$ (so that $\varepsilon(y) \equiv 0$); $\phi^{-1}(y) = y^{1/\rho}$, so that $1-F(x) = e^{-x^{1/\rho}}$. We have (exactly)

$$\mu_n' = \int_0^{\infty} [\phi(x)]^n e^{-y} dy = \frac{1}{\rho} \int_0^{\infty} x^{(1-\rho)/\rho} e^{-x^{1/\rho}} dx = \Gamma(n\rho+1) ;$$

the asymptotic approximation gives

$$\mu_n' \simeq \sqrt{2\pi\, n\rho}\, (n\rho)^{n\rho} e^{-n\rho}$$

which is the same as Stirling's approximation to the gamma function. It is easily seen that $\mu_{n+1}'/\mu_n' \simeq (n\rho)^{\rho}$.

<u>Example 6</u>. $\phi(x) = x^{\rho}/(\ln x)^{\beta}$, with $\rho > 0$, $-\infty < \beta < \infty$; also, $\phi^{-1}(y) \simeq y^{1/\rho}(\ln y)^{-\beta/\rho}$ and $\tau_n \simeq n\rho$. Finally,

$$\mu_{n+1}'/\mu_n' \simeq (n\rho)^{\rho}(\ln n\rho)^{\beta} \simeq (n\rho)^{\rho}(\ln n)^{\beta} .$$

Examples 1 through 6 show that μ_{n+1}'/μ_n' can be asymptotically equal to the following:

$\ldots, \ell_k n, \ell_{k-1} n, \ldots, \ell_2 n = \ln \ln n, \ell_1 n = \ln n,$

$\ldots, n/\ln n, n/\ln \ln n, \ldots, n/\ell_k n, \ldots$

$\ldots, n_1, n^2, \ldots, n^k, \ldots, n^\rho \ (\rho > 0).$

$\ldots, n \ln n, n \ln \ln n, \ldots, n \ell_k n, \ldots$; etc.

We could take $\phi(x) = \exp\{[\ell_k x]^\beta\}, -\infty < \beta < \infty$, if $k \geq 2$, for example.

C. Rapidly Varying Functions.

We let $\phi(x) = \exp[\lambda(x)]$, where $\lambda(x) = x^\rho \exp\{\int_1^x \frac{\varepsilon(y)}{y} dy\}$ with $0 \leq \rho \leq 1$; if $\rho > 1$, the moments μ'_n are infinite for all n. If $\rho = 0$, $\varepsilon(x) \downarrow 0$; if $0 < \rho < 1$, then $\varepsilon(x) \to 0$; and if $\rho = 1$, then $\varepsilon(x) \uparrow 0$. For ϕ varying rapidly, τ_n satisfies $\tau_n = n[\rho+\varepsilon(\tau_n)]\lambda(\tau_n)$;

$\mu'_n \simeq \sqrt{2\pi \tau_n/(1-\rho)}\ [\phi(\tau_n)]^n e^{-\tau_n}$ if $\rho < 1$ and

$\mu'_n \simeq \sqrt{-2\pi \tau_n/\varepsilon(\tau_n)}\ [\phi(\tau_n)]^n e^{-\tau_n}$ if $\rho = 1$, etc. (See 4.8.)

<u>Example 7</u>. $\phi(x) = \exp\{\lambda(x)\}$, with $\lambda(x) = \exp\{\int_1^x \frac{\varepsilon(y)}{y} dy\}$

$$= (\ln x)^\beta, \quad \beta \geq 1.$$

Here, $\varepsilon(x) = \beta/\ln x$ and $\varepsilon(x)\lambda(x) = \beta(\ln x)^{\beta-1}$; τ_n satisfies $\tau_n = n\,\varepsilon(\tau_n)\lambda(\tau_n) = n\,\beta(\ln \tau_n)^{\beta-1}$. To approximate τ_n, define $\tau_{n1} = n\,\varepsilon(n)\lambda(n) = n\,\beta(\ln n)^{\beta-1}$, and, successively, $\tau_{nk} = n[\ln(\tau_{n,k-1})]^{\beta-1}$, $k = 2, 3, \ldots$. It is then easy to show that the sequence $\{\tau_{nk}\}$ is monotonically increasing with $\tau_{nk} < \tau_n$, for all k, and that $\tau_n < \tau_n^* = n\,\beta[\ln n + \ln n/\ln \ln n]^{\beta-1}$. (To show this, we let $f(x) = n\,\varepsilon(x)\lambda(x)/x$. Then $f(\tau_{nk}) > 1$, $f(\tau_n^*) < 1$.) Therefore, in particular, $\tau_{n1} < \tau_n < \tau_n^*$ and

$$1 = \lim_{n\to\infty} \frac{\tau_{n1}}{\tau_{n1}} \leq \lim_{n\to\infty} \frac{\tau_n}{\tau_{n1}} \leq \lim_{n\to\infty} \frac{\tau_n^*}{\tau_1} = 1$$

so that $\tau_n \simeq \tau_{n1}$. Thus, $\tau_{n+1}/\tau_n \to 1$, $\varepsilon(\tau_{n+1})/\varepsilon(\tau_n) \to 1$, and

$$\mu_n' \simeq \sqrt{2\pi\ n\ \beta(\ln n)^{\beta-1}}\ \exp\{n[\ln \tau_{n1}]^{\beta} - \tau_{n1}\}$$

$$\simeq \sqrt{2\pi\ n\ \beta(\ln n)^{\beta-1}}\ \exp\{n(\ln n)^{\beta}\}$$

and $\ln[\mu_{n+1}'/\mu_n'] \simeq \lambda(\tau_n) \simeq \lambda(\tau_{n1}) \sim (\ln n)^{\beta}$.

Finally, we note that $\lambda^{-1}(y) = \exp\{y^{1/\beta}\}$, so that $\lambda^{-1}(\ln x) = \exp\{(\ln x)^{1/\beta}\}$ and hence

$$1\text{-}F(x) = \exp\{-\exp[\ln x]^{1/\beta}\}; \quad \text{if} \quad \beta = 1,\ 1\text{-}F(x) = e^{-x} .$$

Example 8. (See Example 5, Section 1.) Let $1\text{-}F(x) = \exp\{-(\ln x)^2\}$, $x \geq 1$. We evaluate μ_n' directly and then compare it with the asymptotic approximation. We have

$$\mu_n' = n \int_1^{\infty} x^{n-1} \exp\{-(\ln x)^2\}dx = n \int_0^{\infty} (2y)^{-1} \exp\{n\sqrt{y}-y\}dy$$

$$= n \int_0^{\infty} \exp\{n\ z-z^2\}dz = n\ e^{n^2/4} \int_0^{\infty} \exp\{-(z-\tfrac{n}{2})^2\}dz$$

$$= n\sqrt{\pi}\, e^{n^2/4}\ [1 - \Phi(-\frac{n}{\sqrt{2}})]$$

where $\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp\{-z^2/2\}dz$; clearly, $\Phi(-\sqrt{2n}) \to 0$ as $n \to \infty$, so that

$$\mu_n' \simeq n\sqrt{\pi}\, e^{n^2/4} .$$

As for the asymptotic approximation, we have

$$1\text{-}F(x) = e^{-(\ln x)^2} = e^{-\lambda^{-1}(\ln x)} \Rightarrow \lambda^{-1}(x) = x^2 \Rightarrow \lambda(y) = \sqrt{y} .$$

Thus, λ is a regularly varying function of index $\rho = 1/2$, with $\varepsilon(y) \equiv 0$; τ_n satisfies $\tau_n = n[\rho+\varepsilon(\tau_n)]\lambda(\tau_n) = \frac{n}{2}\sqrt{\tau_n}$, so that $\tau_n = n^2/4$; $\lambda(\tau_n) = \sqrt{\tau_n} = n/2$; $n\,\lambda(\tau_n) - \tau_n = n^2/2 - n^2/4 = n^2/4$, and $\sqrt{2\pi\,\tau_n/(1-\rho)} = n\sqrt{\pi}$. Thus, from (2.13), we obtain $\mu_n' \simeq n\sqrt{\pi}\exp\{n^2/4\}$, so that the asymptotic approximation is "asymptotic" to the direct value.

More generally, if $1\text{-}F(x) = \exp\{-(\ln x)^{\beta}\}$, with $\beta > 1$, we find, following the calculations above, that

$$\mu_n' \simeq \sqrt{2\pi\ \beta(n/\beta)^{\beta/(\beta-1)}/(\beta-1)}\ \exp\{n(\beta-1)(n/\beta)^{1/(\beta-1)}/\beta\}$$

and

$$\ln[\mu_{n+1}'/\mu_n'] \simeq \lambda(\tau_n) = (n/\beta)^{1/(\beta-1)} .$$

If we set $\beta = 1 + 1/k$, k a positive integer, then

$$\ln[\mu_{n+1}'/\mu_n'] \simeq [k\ n/(k+1)]^k \simeq (n^k\ e^{-1}) \text{ , for } k \text{ large.}$$

<u>Example 9</u>. Let $\lambda(x) = x/(1+\ln x)^A$, with $A > 0$. Then

$\varepsilon(x) = -A/(1+\ln x)$. In this example, $\rho = 1$ and τ_n satisfies

$$\tau_n = n[1+\varepsilon(\tau_n)]\lambda(\tau_n) = n[1+\varepsilon(\tau_n)]\tau_n/(1+\ln \tau_n)^A$$

so that τ_n satisfies

$$n[1+\varepsilon(\tau_n)]/(1+\ln \tau_n)^A = 1 = n[1 - A/(1+\ln \tau_n)] .$$

(i) Suppose $A = 1$ and let $u_n = 1 + \ln \tau_n$. Then we have

$$n[1 - 1/u_n] \frac{1}{u_n} = 1 \quad \text{or} \quad u_n^2 - n\, u_n + n = 0 ,$$

from which we find that $u_n = [n + \sqrt{n^2-4n}]/2$. Then $\tau_n = \exp\{u_n-1\}$. We next show that $\ln \tau_{n+1}/\tau_n \to 1$, so that $\tau_{n+1}/\tau_n \to L = e$. We have

$$\ln \tau_{n+1}/\tau_n = u_{n+1} - u_n = \{(n+1) + \sqrt{(n+1)^2 - 4(n+1)} - [n + \sqrt{n^2-4n}]\}/2$$

$$= \frac{1}{2} + (2n-3)/2[\sqrt{(n+1)^2 - 4(n+1)} + \sqrt{n^2 - 4n}] \to 1 .$$

Hence,

$$\lambda(\tau_n) = \tau_n/(1 + \ln \tau_n) = u_n^{-1} \exp\{u_n-1\} .$$

Also,

$$\ln g_n(\tau_n) = n\,\lambda(\tau_n) - \tau_n = (n-u_n/u_n) \exp(u_n-1)$$

$$= u_n^{-2} \exp(u_n-1) ,$$

and

$$-\tau_n\ \varepsilon(\tau_n) = u_n^{-2}\exp(u_n-1) \quad\text{and}\quad -\tau_n/\varepsilon(\tau_n) = u_n \exp(u_n-1)\ .$$

Consistent with (2.14) and (2.16), we find that

$$\mu_n' \simeq [-2\pi\ \tau_n/\varepsilon(\tau_n)]^{1/2}\ g_n(\tau_n)$$

$$= 2\sqrt{\pi}\left(u_n\ e^{u_n-1}\right)^{1/2} \exp\{u_n^{-2}\exp(u_n-1)\}$$

and

$$\ln[\mu_{n+1}'/\mu_n'] \simeq -\tau_{n+1}\ \varepsilon(\tau_{n+1}) = (1-e^{-1})u_{n+1}^{-2}\exp(u_n-1)\ .$$

(ii) Suppose that $A \neq 1$. τ_n now satisfies

$$n[1-A/(1+\ln\tau_n]/(1+\ln\tau_n)^A = 1 \equiv n(1-A/u_n)/u_n^A$$

where $u_n = 1 + \ln\tau_n$.

Let $f(x) = n[1 - A/x]/x^A$. Then

$$f(n^{1/A}) = n[1-A/n^{1/A}]\ \frac{1}{n} < 1 \quad \text{(true for all } A > 0)$$

and, for all $\varepsilon > 0$,

$$f[(n-\varepsilon)^{1/A}] = n[1-A/(n-\varepsilon)^{1/A}]/(n-\varepsilon) > 1\ ,$$

if $A < 1$.

Thus, if $A < 1$, $(n-\varepsilon)^{1/A} - 1 < \ln \tau_n < n^{1/A} - 1$. If $A > 1$ and $\varepsilon > 0$ is chosen so that $\varepsilon < 1/A$, then

$$f(n^{1/A-\varepsilon}) = n[1 - A/n^{1/A-\varepsilon}]/n^{1-A\varepsilon} > 1$$

so that for $A > 1$,

$$n^{1/A-\varepsilon} - 1 < \ln \tau_n < n^{1/A} .$$

We can now show that $\tau_{n+1}/\tau_n \to e = L$ if $A > 1$ and $\tau_{n+1}/\tau_n \to \infty$ if $A < 1$, using the inequalities above. More precise bounds on $\ln \tau_n$ are necessary to approximate μ'_n and $\ln(\mu'_{n+1}/\mu'_n)$ - we content ourselves here to have shown that it is possible for $\tau_{n+1}/\tau_n \to L > 1$ *finite and* $\tau_{n+1}/\tau_n \to \infty$.

Example 10. Let $\lambda(x) = x \exp\{-(1 + \ln x)^{1/2}\}$. Then $\varepsilon(x) = -\frac{1}{2}(1 + \ln x)^{-1/2}$, and τ_n satisfies

$$n[1 - 1/2V_n]\, e^{-V_n} = 1$$

where $V_n = \sqrt{1 + \ln \tau_n}$. Let $f(x) = n[1 - \frac{1}{2x}]e^{-x}$. then

$$f(\ln n - \frac{1}{2\ln n}) < 1 \quad \text{and} \quad f(\ln n - \frac{1}{\ln n}) > 1 \quad ,$$

so that

$$\ln n - \frac{1}{\ln n} < V_n < \ln n - \frac{1}{2\ln n}$$

or

$$(\ln n - \frac{1}{\ln n})^2 - 1 < \ln \tau_n < (\ln n - \frac{1}{2\ln n})^2 - 1 .$$

Using this inequality, it can be shown that $\tau_{n+1}/\tau_n \to 1$.

Summarizing, Examples 9 and 10 demonstrate that τ_{n+1}/τ_n can converge to 1, e, and ∞, if $\rho = 1$. Whether or not other limiting values are possible is unknown; we conjecture that these are the only ones possible.

Example 11. Recalling that $\ell_1 x = \ln x$, $\ell_2 x = \ln \ln x$, etc., and $e_1 x = e^x$, $e_2 x = \exp(e^x)$, etc., we let $\lambda(x) = \ell_k x$, for $x \geq e_k(1)$. Then

$$\varepsilon(x) = -[(\ell_1 x)(\ell_2 x)\ldots(\ell_k x)]^{-1}$$

and τ_n satisfies the equation

$$n[1+\varepsilon(\tau_n)]\lambda(\tau_n) = n[1 - \{(\ell_1 \tau_n)(\ell_2 \tau_n)\ldots(\ell_k \tau_n)\}^{-1}]/\ell_k \tau_n = 1 .$$

Letting $f(x) = n[1+\varepsilon(x)]/\ell_k x$, it is not hard to show that

$$f(\ell_k(n)) < 1 \quad \text{and} \quad f[\ell_k\{n[1+\varepsilon(n)]\}] > 1$$

so that

$$\tau_{n1} = e_k\left\{n\left[1 - \frac{1}{(\ell_1\ n)(\ell_2\ n)\ldots(\ell_k\ n)}\right]\right\} < \tau_n < e_k(n)\ .$$

In fact, if we define $\tau_{n2} = e_k\ n[1+\varepsilon(\tau_{n1})]$, ..., $\tau_{nk} = e_k\{n[1+\varepsilon(\tau_{n,k-1})]\}$, then the sequence $\{\tau_{nk}\}$ is monotonically increasing with $\tau_{nk} < e_k(n)$ for all k ; this fact can be used to approximate τ_n arbitrarily closely. The inequalities above are, however, sufficient for our purpose here, which is to provide a lower bound for μ_n' and $-\tau_{n+1}\ \varepsilon(\tau_{n+1})$. We have

$$\ln g_n(\tau_n) = n\ \lambda(\tau_n) - \tau_n = \tau_n[(n/\ell_k\ \tau_n) - 1] = \tau_n[n - \ell_k\ \tau_n]/\ell_k\ \tau_n$$

$$\geq \tau_{n1}\ n - n[1 - \{(\ell_1\ n)(\ell_2\ n)\ldots(\ell_k\ n)\}^{-1}]\}$$

$$= \tau_{n1}/[(\ell_1\ n)(\ell_2\ n)\ldots(\ell_k\ n)]\ .$$

Also, $-\tau_n/[\varepsilon(\tau_n)] > \tau_{n1}[(\ell_1\ \tau_{n1})(\ell_2\ \tau_{n1})\ldots(\ell_k\ \tau_{n1})]$ and

$$-\tau_n\ \varepsilon(\tau_n) > \tau_{n1}/[(\ln n)(\ln \ln n)\ldots(\ell_k\ n)]\ .$$

Thus,

$$\mu_n' \simeq \sqrt{-2\pi\ \tau_n/\varepsilon(\tau_n)}\ g_n(\tau_n)$$

$$> \sqrt{2\pi\ \tau_{n1}[(\ell_1\ \tau_{n1})\ldots(\ell_k\ \tau_{n1})]}\ \tau_{n1}/[(\ell_1\ n)\ldots(\ell_k\ n)]$$

and

$$\ln[\mu'_{n+1}/\mu'_n] \simeq -\tau_{n+1}\, \varepsilon(\tau_{n+1}) > \tau_{n1}/[(\ell_1\ n)(\ell_2\ n)\ldots(\ell_k\ n)] \ .$$

For example, if $k = 3$, we have, explicitly,

$$\ln[\mu'_{n+1}/\mu'_n] \quad \frac{e^{e^{e^{n[1 - \frac{1}{(\ln n)(\ln \ln n)(\ln \ln \ln n)}]}}}}{(\ln n)(\ln \ln n)(\ln \ln \ln n)} \ .$$

6. Some Comments.

Additional examples of growth rates may be constructed, noting the following. Suppose Y_i, $i = 1,\ldots,k$ are random variables with distribution functions F_{Y_i} such that the moments of F_{Y_i} have growth rates $\mu'_{n+1}/\mu'_n \sim \phi_i(\tau_n)$. Then the random variable $Y = \Pi_{i=1}^k Y_i$ has moments $\mu_{n+1}^{(Y)}/\mu_{n'}^{(Y)} \simeq \Pi_{i=1}^k \phi_i(\tau_{ni})$ if the $\{Y_i\}$ are independent. For example, let $1-F_{Y_1}(y) = \exp\{-e^y\}$, corresponding to having $\mu'_{n+1}/\mu'_n \simeq \ln n = \phi_1(\tau_n)$, and $1-F_{Y_2}(y) = \exp\{-(\ln y)^2\}$, leading to $\mu'_{n+1}/\mu'_n \simeq \exp\{2n+1\}$. Then $Y = Y_1\ Y_2$ has moments such that $\mu'_{n+1}/\mu'_n \simeq (\ln n)\ \exp\{e^{2n+1}\}$.

Distribution functions F which have moments which grow "relatively" fast are frequently cited as ones where the moment sequence $\{\mu'_n\}$ fails to determine F uniquely. For example, the distribution function F in Example 5 of Section 5 when $\rho = 1/4$ has moments $\mu'_n = \Gamma(4n+1)$. It is known that this sequence does not determine F uniquely - it can be shown, for example, that

$$\int_0^\infty x^n \exp\{-x^{1/4}\}dx = \int_0^\infty x^n[1 - \sin(x^{1/4})]\ \exp\{-x^{1/4}\}dx \ .$$

and $1-F^{*}(x) = [1-\sin(x^{1/4})]\exp\{-x^{1/4}\}$ corresponds to a distribution function having the same moments as F. Furthermore, if $1-F(x)$ is such that $1-F(x) \geq \exp\{-x^{1/4}\}$, then F cannot be determined by its moments since we can then add $[1-\sin(x^{1/4})]\exp\{-x^{1/4}\}$ to $1-F(x)$ to get another distribution function having the same moments as F. Thus, if ϕ is a monotone increasing regularly varying function such that ϕ^{-1} (which is also necessarily at least slowly varying) has index of regular variation $0 \leq \rho \leq 1/4$ or is slowly varying $(\rho = 0)$, then $1-F(x) = \phi^{-1}(x)$ cannot be uniquely determined by its moments. For example, if $\phi(x) = \exp\{\lambda(x)\}$, with $\lambda(x) = x^{\rho} L(x)$, $0 \leq \rho \leq 1$, then $1-F(x) = \exp\{-\phi^{-1}(x)\}$, is not determined by its moments. Thus, we have introduced here a class of distributions not determined by its moments, thereby explicitly adding to the existing store of such examples.

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GROWTH RATES OF MOMENT SEQUENCES

General properties of growth rates of moment sequences $\{\mu_n'\}$ of non-negative random variables are presented. Then asymptotic results on moment sequences are derived for two classes of distribution functions. Explicitly, let ~~ϕ be~~ g a monotone l.c. increasing twice differentiable regularly varying function at infinity, ~~with~~ $\phi(+\infty) = +\infty$ and index of variation $\rho \geq 0$. If we define the distribution function F by $-\ln[1 - F(x)] = \phi^{-1}(x)$, then it is shown that the n^{th} moment μ_n' and the ratio μ_{n+1}'/μ_n' are such that

$$\mu_n' = \int_0^\infty [\phi(y)]^n e^{-y} dy \text{ and } \mu_{n+1}'/\mu_n' \simeq \phi(\tau_n),$$

where τ_n satisfies the equation $n\phi'(\tau_n)/\phi(\tau_n) = 1$. A second class of distributions functions is defined, ~~by setting~~ $-\ln[1 - F(x)] = \phi^{-1}(\ln x)$, where now $\rho \leq 1$. In this case, the n^{th} moment of F is given by $\mu_n' = \int_0^\infty e^{n\phi(y) - y} dy$; the asymptotic behavior of μ_{n+1}'/μ_n' is determined, also. Finally, several examples are given to illustrate the possible different asymptotic growth rates of moments.